CONFORMAL STRUCTURES AND HARMONIC FUNCTIONS

SIDDHARTHA GADGIL AND HARISH SESHADRI

Abstract. We study conformal structures in terms of the kernel of the conformal Laplacian. Our main goal is to build a common framework to study both conformal classes of Riemannian metrics and degenerate conformal structures, in particular Carnot-Caratheodary structures (as also piecewise-flat conformal structures).

We introduce curvatures and torsions associated to sheafs of functions that determine when the sheaf is the kernel of the conformal Laplacian for a Riemannian metric and when the metric is conformally flat.

A conformal structure on a smooth manifold M is an equivalence class of Riemannian metrics on M, where two metrics g and g' are equivalent if for some positive smooth function $f : M \to \mathbb{R}^+$, $g' = f^2 g$. Our goal here is to give an alternative characterisation of a conformal structure, in terms of the kernel of the conformal Laplacian, and use this to study various notions of conformal geometry in a manner that applies also to more geeneral cases including Carnot-Caratheodary and PL-metrics. We begin by outlining our characterisation.

Given a Riemannian metric g on a manifold M, we consider for every open set U of M the kernel of the conformal Laplacian on U (for definitions see Section **??**). The projective class of this sheaf of functions depends only on the conformal class of g. We show conversely in section **??** that the sheaf of functions in turn determines the conformal class of g. Thus, we can characterise conformal structures in terms of these functions, which we call $\Box$-harmonic. The main technical result needed is the existence and C^2-control of the solutions for the conformal Laplacian on a small domain. Sheaves of functions constructed as above satisfy certain elementary properties. These are axiomatised in section **??** for use on the later sections.

Perhaps the most interesting result of this paper is a characterisation of a conformally flat structure in terms of a sheaf of functions satisfying certain axioms (given in section **??**). The basic idea is that conformally flat structures admit conformal homotheties, and hence sheafs coming from metrics that are conformally flat admit homotheties that are conformal. More generally, one can consider homotheties that are conformal "up to a certain order". The obstruction to existence of such homotheties gives curvatures of the corresponding order.

As all Riemannian metrics are flat up to first order, we obtain *first-order curvatures* that are the obstructions to a sheaf of functions being the sheaf of $\Box$-harmonic functions for a Riemannian metric. If these as well as an invariant which we call the *torsion* vanish then the sheaf corresponds to a metric. The torsion measures the relation between homotheties at different points. We further obtain *conformal curvatures* that determine whether the metric is conformally flat.

Analogous curvatures can be constructed in the case of Carnot-Caratheodary metrics. In the case of a Carnot metric associated to a contact structure, the

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model for a flat conformal structure is the boundary of complex hyperbolic space. More generally, the boundaries of rank 1 locally symmetric spaces are models for the Carnot metrics associated to appropriate distributions. The analogues of our results for these cases should provide a single framework to study the boundaries of rank 1 locally symmetric spaces. It is conjectured in [?] that a C^2-structure (compatible with the fundamental group action) on the boundary of the universal cover of a negatively curved manifold implies that it is locally symmetric. Since a C^2-structure is what is needed to construct the sheafs we consider, this approach may also be relevant to the conjecture mentioned above.

Note that Pansu [?] has a different notion of conformal structures that is more general but does not have notions of curvature as we do.

The other advantage of our approach is in cases where we can construct a sheaf of $\Box$-harmonic functions even though there is no underlying Riemannian or Carnot metric (at least *a priori*). In this paper we consider two such applications. The first is to study the conformal geometry of piecewise-linear metrics with respect to a triangulation. A PL-metric gives rise to a piecewise conformal structure in our sense, though there is no classical analogue for this.

Our second application is to consider convergence and degeneration of conformal structures. In Riemannian geometry, generally the difficulty in studying this lies in *choosing* metrics in each conformal class in the sequence appropriately, typically with some control on the curvature. From our point of view, however, one can simply take the weak limit of the sheaf of $\Box$-harmonic functions. The resulting sheaf may not correspond to a Riemannian metric. In this case we get a *degenerate* conformal structure, i.e., a conformal structure corresponding to a Carnot metric. We shall study such degenerations in Section **??**. In this paper we provide a framework to understand such degenerations, which we hope will have applications in the future.

The organisation of the paper is as follows: In Section **??**, we show that there is a bijective correspondence conformal classes of Riemannian metrics and sheafs of $\Box$-harmonic functions. In Section **??**, we axiomatize the properties of sheafs which arise as sheafs of $\Box$-harmonic functions. The purpose here is the study of conformal geometry even in the absence of a Riemannian metric. The exact characterisation of a conformal structure coming from a Riemannian metric is explored further in Section **??**. In particular, we show that the obstruction to this lies in what we call *first-order curvatures* and an invariant we call the *torsion*. When these vanish, we see that the obstruction to conformal flatness can be recovered from the sheaf. Sections **??** and **??** deal with the application of these ideas to two cases. In Section 5, we define conformal structures associated to piecewise-Euclidean metrics on triangulated manifolds. The $\Box$-harmonic functions here will be those that are $\Box$-harmonic away from the codimension-2 skeleton. This definition is motivated by the fact that sets of codimension 2 are polar and hence whether a function is harmonic is determined by its restircition to the complement of the two skeleton. In Section 6 we consider sequences of conformal structures and show that one gets a limit conformal structure under fairly weak hypotheses. Finally, in section 7, we make some speculative remarks.

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1. Preliminaries

1.1. **The Conformal Laplacian.** For the rest of the paper, "smooth" will mean C^∞. Let M be a smooth n-manifold and g a Riemannian metric on M. By the Laplacian of g we mean the Laplacian acting on functions given by

$$\triangle f = -d^* df,$$

for any smooth function f on M. We will write $\triangle_g$, instead of $\triangle$ if we want to emphasize the dependence on g.

When $n = 2$ and u is a smooth positive function, we know that $\triangle_{ug} = u^{-1}\triangle_g$. When $n \geq 3$, the *conformal Laplacian* of g is the differential operator

$$\Box_g = -\frac{4(n-1)}{n-2}\triangle_g + s_g,$$

where s_g is the scalar curvature of g. This operator is *conformally invariant* in the following sense: if $\tilde{g} = u^{\frac{4}{n-2}}g$ for u as above, then

$$\Box_{\tilde{g}} f = u^{-\frac{n+2}{n-2}}\Box_g uf$$

for any function f. In particular

$$Ker\Box_{\tilde{g}} = u^{-1}Ker\Box_g.$$

A function in $Ker\Box_g$ will be called a $\Box$-*harmonic function.* The above relations show that the projective class of $\Box$-harmonic (harmonic, when $n = 2$) functions depends only on the conformal class of g.

The conformal invariance of $\Box$ follows from the transformation laws for the Laplacian and scalar curvature under conformal changes (see, for instance [**?**], Page 59). For scalar curvature, we have

$$u^{\frac{n+2}{n-2}}s_{\tilde{g}} = -4\frac{n-1}{n-2}\triangle_g u + s_g u = \Box_g u.$$

In particular, we have the following simple fact.

Proposition 1.1. *If u is positive and $\Box_g$-harmonic, then $\tilde{g} = u^{\frac{4}{n-2}}g$ has zero scalar curvature.*

1.2. **Elliptic Estimates.** Let $\Omega \subset \mathbb{R}^n$ be an open domain of diameter $\leq D$ and smooth boundary. Let u a smooth function on Ω.

For the first estimate, we look at linear elliptic differential operators of the following form:

$$Lu = a^{ij}\partial_i\partial_j u + b^i\partial_i u,$$

where $a^{ij} = a^{ji}$ and b^i are smooth functions on Ω and $\partial_i = \frac{\partial}{\partial_i}$. Suppose that all eigenvalues of (a^{ij}) lie in $[\lambda, \lambda^{-1}]$ and $max\ \{\|a^{ij}\|_{C^\alpha,\Omega}, \|a^{ij}\|_{C^\alpha,\Omega}\} \leq \lambda$.

Theorem 1.2. ([**?**] Theorem 2.2, Page 284) *If $u = \phi$ on $\partial\Omega$, then, for any $\alpha \in (0,1)$, there is a constant $C = C(n, \alpha, \lambda, D)$ such that*

$$\|u\|_{C^{2,\alpha},\Omega} \leq C(\|Lu\|_{C^\alpha,\Omega} + \|\phi\|_{C^{2,\alpha},\partial\Omega}).$$

Note that the Laplacian of any Riemannian metric is of the required form since

$$\triangle u = \frac{1}{\sqrt{detg_{ij}}}\partial_i(\sqrt{detg_{ij}}g^{ij}\partial_j u)$$

in local coordinates.

The second estimate is for general elliptic operators. Let L be given by

$$Lu = a^{ij}\partial_i\partial_j u + b^i\partial_i u + cu.$$

Assume that that the least eigenvalue of (a^{ij}) is at least λ and for a given $\alpha \in (0,1)$, $max\ \{\|a^{ij}\|_{C^\alpha,\Omega}, \|b^i\|_{C^\alpha,\Omega}, \|c\|_{C^\alpha,\Omega},\} \leq \Lambda$.

Theorem 1.3. ([?] Theorem 6.6) *If ϕ is a smooth function on Ω and $u = \phi$ on $\partial\Omega$, then, for any $\alpha \in (0,1)$, there is a constant $C = C(n,\alpha,\lambda,\Lambda,\Omega)$ such that*

$$\|u\|_{C^{2,\alpha},\Omega} \leq C(\|u\|_{C^0,\Omega} + \|Lu\|_{C^\alpha,\Omega} + \|\phi\|_{C^{2,\alpha},\Omega}).$$

Here the dependence of C on Ω is through the geometry of the boundary $\partial\Omega$.

Finally, we state an *a priori* estimate for domains of small diameter. L is a general elliptic operator as in the previous theorem. Let $\beta = sup\frac{\|B\|}{\lambda}$, where $B = (b^1,...,b^n)$, $C = e^{(\beta+1)d} - 1$, where d is the diameter of Ω and $C_1 = 1 - Csup\ _\Omega\frac{c^+}{\lambda}$.

Theorem 1.4. ([?] Corollary 3.8) *If $C_1 > 0$, then*

$$sup\ _\Omega|u| \leq \frac{1}{C_1}(sup\ _{\partial\Omega}|u| + Csup\ _\Omega\frac{|Lu|}{\lambda}).$$

1.3. **Harmonic Coordinates.** We will use harmonic coordinates at various points in this paper. The following proposition guarantees their existence.

Proposition 1.5. [?] *If (M,g) is a Riemannian n-manifold and $p \in M$, there is a neighbourhood U containing p on which we can find a harmonic coordinate system $x = (x_1,...,x_n) : U \to \mathbb{R}^n$, i.e., a coordinate system such that the functions x_i satisfy $\triangle x_i = 0$. Similarly, there exists a coordinate system y_i with $\Box y_i = 0$.*

1.4. **Carnot-Caratheodary metrics.**

Definition 1.6. A Carnot-Caratheodary metric of codimension-k on a manifold M consists of a smooth codimension-k distribution $\xi_p \subset T_pM$, $p \in M$ of TM and for each p a positive definite inner product on ξ_p varying smoothly with p.

A codimension 0 Carnot-Caratheodary metric is just a Riemannian metric. In general, a Carnot-Caratheodary metric induces a length function on smooth paths γ that are tangent to ξ. This induces (as in the Riemannian case) a distance funtion on M which however may be infinite for some pairs of points.

For brevity, we sometimes call a Carnot-Caratheodary metric a Carnot metric.

2. Differential operators and conformal structures

Let M be a smooth manifold and g a Riemannian metric on M. We denote by $\mathfrak{H}$ the sheaf of $\Box$-harmonic functions. Note that under a conformal change of metric $g \to \tilde{g} = u^{\frac{4}{n-2}}g$, $\mathfrak{H}$ changes to $u^{-1}\mathfrak{H}$. Hence we have the following proposition.

Proposition 2.1. *A conformal class of Riemannian metrics on determines a projective class of sheafs of functions on M.*

The main content of this section is that the converse of this is true. We can in fact recover the conformal structure from the 2-jets of $\Box$-harmonic functions. This follows as, as we show below, two co-vectors u and v in $T^*_p(M)$ are perpendicular if and only if there are functions f and g on a neighbourhood of p with $f(p) = g(p) = 0$, $df(p) = u$, $dg(p) = v$ and with fg $\Box$-harmonic.

Considering 2-jets of $\Box$-harmonic functions as above is equivalent to simply considering projective classes of inner-products on the tangent space. However considering the sheaf of $\Box$-harmonic functions (rather than projectivised inner products) is useful in many contexts, for instance we get a criterion for conformal flatness as also a natural notion of piece-wise conformal structures.

We first need a technical lemma relating the 2-jet of $\Box$-harmonic functions to global $\Box$-harmonic functions.

Lemma 2.2. *Let $p \in M$ be a point. Suppose f is a function defined on a neighbourhood of p, then $\triangle f(p) = 0$ (respectively $\Box f(p) = 0$) if and only if there is a function $g : U \to R$ defined on a neighbourhood of p with $\triangle g = 0$ (respectively $\Box g = 0$) on U and with $f = g$ up to second order near p.*

Proof. We first prove the result for $\triangle$ and then indicate the changes needed to deduce the result for $\Box$.

Let $V \subset \mathbb{R} \oplus T_p^*M \oplus T_pM \otimes T_p^*M$ be the subspace defined by

$$V = \{(c, \psi, \Psi) \ : Trace\Psi = 0\}$$

and let W be the subspace of V consisting of 2-jets of $\triangle$-harmonic functions defined in neighbourhoods of p, i.e. the set of $(f(p), df(p), D^2f(p))$ with $\triangle f = 0$. Here $D^2f(p)$ denotes the Hessian of f at p, regarded as an element of $T_pM \otimes T_p^*M$.

Given $v = (f(p), df(p), D^2f(p)) \in V$, we want to prove that $v \in W$. Suppose $d(v, W) > 0$, where d denotes distance. Let ϕ_ϵ be the solution to the following Dirichlet problem on B_ϵ (B_ϵ denotes the ball of radius ϵ at p) :

$$\triangle\phi_\epsilon = 0 \text{ on } B_\epsilon, \quad \phi_\epsilon = f \quad \text{on } \partial B_\epsilon.$$

The Schauder estimate of Theorem **??** gives

$$\|\phi_\epsilon - f\|_{C^{2,\alpha},B_\epsilon} \leq C\|\triangle f\|_{C^\alpha,B_\epsilon}.$$

Here $C = C(\epsilon)$ is bounded above as long as ϵ is bounded above. Hence, as $\epsilon \to 0$, $\|v - (\phi_\epsilon(p), d\phi_\epsilon(p), D^2\phi_\epsilon)\| \to 0$. This contradicts our assumption that $d(v, W) > 0$.

Next consider the case of $\Box$-harmonic functions. Let f satisfy $\Box f(p) = 0$. We want to construct a $\Box$-harmonic function ψ with the same 2-jet as f at p. We now take the subspaces V and W as follows:

$$V = \{(c, \psi, \Psi) \ : Trace\Psi = s(p)c\}$$

and W to be the subspace of V consisting of 2-jets at p of $\Box$-harmonic functions. We have to show first that the Dirichlet problem for $\Box$ can be solved on small balls. This immediately follows from the Fredholm alternative of [**?**] Theorem 6.15, i.e., to prove the existence of solutions to

$$\Box u = 0 \text{ on } B_\epsilon, \quad u = f \text{ on } \partial B_\epsilon,$$

it is enough to prove that the only solution to

$$\Box u = 0 \text{ on } B_\epsilon, \quad u = 0 \text{ on } \partial B_\epsilon$$

is the trivial one.

So let u satisfy the latter equations. Then

$$a_n \int_{B_\epsilon} u\triangle u = \int_{B_\epsilon} -su^2,$$

where $a_n = -4\frac{n-1}{n-2}$ which implies that

$$a_n \lambda_1(\epsilon) \leq \frac{\int_{B_\epsilon} u(-\triangle u)}{\int_{B_\epsilon} u^2} \leq Sup\ _{B_\epsilon}|s|,$$

where $\lambda_1(\epsilon)$ is the first eigenvalue of $\triangle$ with Dirichlet boundary conditions on B_ϵ. It is well-known (see [?], Page 318) that $\lambda_1(\epsilon) \to -\infty$ as $\epsilon \to 0$. Hence, if ϵ is sufficiently small, we get a contradiction, since $a_n < 0$, unless $u \equiv 0$.

To continue with the proof, we let ψ_ϵ be the solution to

$$\Box \psi_\epsilon = 0 \text{ on } B_\epsilon, \quad \psi_\epsilon = f \text{ on } \partial B_\epsilon,$$

where $\Box f(p) = 0$.

Here we have to use the Schauder estimate for general differential operators given in Theorem **??**. Applying it to $\psi_\epsilon - f$, we get

$$\|\psi_\epsilon - f\|_{C^{2,\alpha},B_\epsilon} \leq C(\|\psi_\epsilon - f\|_{C^0,B_\epsilon} + \|\Box f\|_{C^\alpha,B_\epsilon}),$$

where the constant C remains bounded as ϵ goes to zero. In order to argue as earlier we need to show that

$$\|\psi_\epsilon - f\|_{C^0,B_\epsilon} \to 0$$

as $\epsilon \to 0$. This follows from the *a priori* bound of Theorem **??** for solutions of $Lu = h$, where L is a linear elliptic differential operator on domains of small diameter. In our case we take $L = \Box$, $h = \Box f$ and $u = \psi_\epsilon - f$.

$$sup_{B_\epsilon}|\psi_\epsilon - f| \leq D\ sup_{B_\epsilon}|\Box f|.$$

Here we have used the fact that $\psi_\epsilon - f = 0$ on ∂B_ϵ. From the reference quoted above we note that $D = D(\epsilon)$ goes to zero as ϵ does. Either from this fact or from $\Box f(p) = 0$, we see that

$$\|\psi_\epsilon - f\|_{C^0,B_\epsilon} \to 0$$

as $\epsilon \to 0$. The rest of the proof is as in the case of $\triangle$.

□

Corollary 2.3. *Any Riemannian metric is locally conformally equivalent to one with zero scalar curvature.*

Proof. This follows from Proposition **??** and the above. □

Now let $p \in M$ be a point. Let $\mathcal{J}_p$ be the equivalence classes of smooth functions vanishing at p, where two functions are equivalent if they agree up to second order near p.

Remark 2.4. Suppose $f(p) = g(p) = 0$. Then $\triangle f(p) = 0$ if and only if $\Box f(p) = 0$ and if $f \sim g$, $\triangle f(p) = 0$ if and only if $\triangle g(p) = 0$.

We now show how the conformal structure can be recovered from the $\Box$-harmonic functions.

Lemma 2.5. *Two non-zero co-vectors u and v in $T_p^*(M)$ are perpendicular if and only if there are functions f and g on a neighbourhood of p with $f(p) = g(p) = 0$, $df(p) = u$, $dg(p) = v$ and with fg $\Box$-harmonic.*

Proof. Let f and g be functions with $df(p) = u,\ dg(p) = v$ and $f(p) = g(p) = 0$. It is clear from the equation

$$\triangle fg = f\triangle g + g\triangle f + < df, dg >$$

that $\triangle fg(p) = 0$ if and only if u and v are perpendicular.

Now,$\triangle fg(p) = 0$ if and only if $\square fg(p) = 0$. By lemma **??**, $\square fg(p) = 0$ if and only if $fg \equiv \phi$ with ϕ $\square$-harmonic in an neighbourhood of p. Thus, if u and v are perpendicular, we can replace g by $g' = \phi/f$. It is easy to see that g' is smooth at p and, in fact, $dg' = dg = v$. Hence f and g' are as required. The converse is immediate. $\square$

From this, we can conclude that the conformal class of the metric is determined by the projectivised sheaf of $\square$-harmonic functions.

Theorem 2.6. *The conformal class of a metric g is determined by the projective class of $\mathfrak{H}$.*

Proof. Given $\mathfrak{H}$, the above lemma allows one to recover when two vectors in T^*M are perpendicular, hence giving the conformal structure. Next note that by the above it suffices to know, for each point p, the harmonic functions f with $f(p) = df(p) = 0$ up to second order. This class is not changed by multiplying the sheaf $\mathfrak{H}$ by a positive smooth function. $\square$

Recall that a diffeomorphism ϕ of the Riemannian manifold (M, g) is called *conformal* if $\phi^* g = u^{\frac{4}{n-2}} g$ for some positive function u. We will also refer to a conformal diffeomorphism as as a conformal *map*.

Corollary 2.7. *ϕ is a conformal diffeomorphism of (M, g) if and only if $\phi^*\mathfrak{H} = u^{-1}\mathfrak{H}$ for some smooth positive function u.*

Proof. This follows immediately from the preceding theorem and the fact that $\phi^*\mathfrak{H} = \mathfrak{H}_{\phi^*(g)}$. $\square$

We shall now obtain a more precise local description of harmonic functions. Let $Q(p)$ be the subset of $\mathcal{J}_p$ consisting of equivalence classes of functions f with $f(p) = df(p) = 0$ and $H(p)$ be the subset of $Q(p)$ consisting of those equivalence classes that contain $\square$-harmonic functions.

Proposition 2.8. *$H(p)$ is spanned by equivalence classes of the form $fg \in Q(p)$ with $df(p)$ perpendicular to $dg(p)$.*

Proof. We have already seen that equivalence classes of fg as above are in $H(p)$. Conversely, let x_i be harmonic functions with $u_i = dx_i(p)$ an orthonormal basis of $T^*(p)$. Then $Q(p)$ has a basis consisting of the equivalence classes of $\phi_{ij} = x_i x_j$, those of $\psi_{ij} = x_i^2 - x_j^2$ and that of $\Phi = \Sigma x_i^2$. By the above, using the identity $x_i^2 - x_j^2 = \frac{(x_i+x_j)(x_i-x_j)}{2}$, all the functions in this basis except Φ are of the form fg with $df(p)$ and $dg(p)$ perpendicular. On the other hand $\triangle\Phi(p) \neq 0$. Thus $H(p)$ is spanned by functions of the form fg with $df(p)$ and $dg(p)$ perpendicular. $\square$

Remark 2.9. Given a choice of metric, a function $\Phi_p \in Q(p)$ is canonically defined as above. For a conformally equivalent metric the corresponding function is a scalar multiple of Φ_p. Thus, the subspace spanned by Φ_p is determined by the conformal structure.

In harmonic co-ordinates, x_i for a metric g, we obtain a precise description.

Proposition 2.10. *The classes of harmonic function in $\mathcal{J}_p$ are those represented by a function of the form $f(x) = c + \Sigma a_i x_i + q(x)$ with $q(x) \in H(p)$.*

Proof. Any class in $\mathcal{J}_p$ has a representative of the form $f(x) = c + \Sigma a_i x_i + q(x)$ with $q(x) \in Q(x)$. As $c + \Sigma a_i x_i$ is harmonic, f is equivalent to a harmonic function if and only if $q(x) \in H(p)$. □

Remark 2.11. Even though the sheaf $\mathfrak{H}$ is defined only projectively, the space $H(p) \subset Q(p)$ is well-defined for a given conformal class.

3. Axiomatics

In this section we study further the connection between sheafs of functions and conformal structures. While everything is stated for smooth manifolds, we have stated the definitions in a manner that readily generalises to varieties over any field with (formal) power series taking the place of smooth functions.

3.1. **Regularity.** Suppose we are given a sheaf of functions $\mathfrak{H}$ on M. For this to meaningfully correspond to a conformal structure at a point $p \in M$, we need some regularity conditions. Recall that $Q(p)$ consists of equivalence classes of functions that vanish up to first order at p, with two functions equivalent if they agree up to second order.

Definition 3.1. We say that $\mathfrak{H}$ is regular at p if the following hold.

(1) There is a function Ψ in $\mathfrak{H}(U)$ for a neighbourhood U of p with $\Psi(p) \neq 0$.
(2) Given u in $T_p^*(M)$, there is a function f in $\mathfrak{H}(U)$ for a neighbourhood U of p with $f(p) = 0$ and $df(p) = u$.
(3) The subspace $H(p)$ of $Q(p)$ consisting of equivalence classes of functions in $\mathfrak{H}(U)$ for neighbourhoods U of p has codimension 1.

Observe that for the sheaf corresponding to a metric every point is regular by Lemma **??**. We say that p is *almost regular* if the above is satisfied except that we allow $H(p) = Q(p)$ instead of the third condition.

Proposition 3.2. *If $\mathfrak{H}$ is regular at p, then $\mathfrak{H}$ is almost regular at every point q in a neighbourhood of p.*

Proof. For q sufficiently close to p, $\Psi(q) \neq 0$. Further, if f_i are functions such that $f_i(p) = 0$ and $\{df_i(p)\}$ form a basis of $T_p^*(M)$, then $g_i = f_i - \frac{f_i(q)}{\Psi(q)}\Psi$ satisfy $g_i(q) = 0$ and $\{dg_i(q)\}$ span $T_q^*(M)$. Similarly, we can conclude that $H(q)$ has codimension at most one. □

3.2. **Pointwise structure.** For a regular point $p \in M$, we construct a symmetric bilinear form on $T_p^*(M)$. Observe as $H(p)$ has codimension one in $Q(p)$, we have a canonically defined (up to scaling) homomorphism $q : Q(p) \to \mathbb{R}$ that has kernel $H(p)$. Pick such a q.

Now suppose $u, v \in T_p^*(M)$ are two co-vectors. Let f and g be functions defined on a neighbourhood of p with $f(p) = g(p) = 0$, $df(p) = u$, $dg(p) = v$. We define the bilinear form ϕ on $T_p^*(M)$ by $\phi(u, v) = q([fg])$ where $[fg]$ is the equivalence class in $Q(p)$ and depends only on u and v.

In general, the resulting form may not be positive definite. If it is negative definite then replacing q by $-q$ leads to a positive definite form. The following characterise when we get a positive definite form, after possibly making this change.

Definition 3.3. The sheaf $\mathfrak{H}$ is *definite* at a regular point p if for all smooth functions f with $f(p) = 0$ and $df(p) \neq 0$, $[f^2] \notin H(p)$.

When we consider sequences of conformal structures, the sheaf often degenerates to give one corresponding to a Carnot metric. The corresponding sheafs can be characterised as below.

Definition 3.4. The sheaf $\mathfrak{H}$ is *semi-definite* at a regular point p if for all smooth functions f, g with $f(p) = g(p) = 0$ and $df(p) \neq 0 \neq dg(p)$,

$$[f^2 + g^2] \in H(p) \implies [f^2] \in H(p)$$

3.3. **Local determination.** A function is $\Box$-harmonic if it is $\Box$-harmonic at every point. Further, whether it is $\Box$-harmonic at a point depends only on its 2-jet. We shall say that the sheaf $\mathfrak{H}$ is locally determined if it satisfies the analogue of these properties.

Definition 3.5. A smooth function f is said to be *harmonic* with respect to $\mathfrak{H}$ at a regular point p in M if for $\Phi \in \mathfrak{H}$ with $f(p) = \Phi(p)$ and $u \in \mathfrak{H}$ with $u(p) = 0$ and $du(p) = df(p) - d\Phi(p)$, $[f - \Phi - u] \in H_p$.

Note that Φ and u exist by regularity and the condition does not depend on the choice of Φ and u. Henceforth assume that the sheaf $\mathfrak{H}$ is regular at every point.

Definition 3.6. The sheaf $\mathfrak{H}$ is *locally determined* if for every smooth function f on $U \subset M$, $f \in \mathfrak{H}(U)$ if and only if f is harmonic at p for all $p \in U$.

Thus, sheafs $\mathfrak{H}$ associated to Riemannian metrics are regular at all points and locally determined. It is not however true that all such sheafs come from a metric. This follows by considering the relation between $\mathfrak{H}$ and the H_p.

Proposition 3.7. *Regular, locally determined sheafs $\mathfrak{H}$ near a point p are determined by $H(q)$ for q near p together with functions Φ and f_i defined near p with $\Phi(p) \neq 0$, $f_i(p) = 0$ and $df_i(p)$ forming a basis for $T_p^*(M)$.*

Proof. As in the proof of Proposition **??**, we can construct g_i for q near p with $g_i(q) = 0$ and dg_i spanning the cotangent space. Thus, for any function f there is a unique linear combination $u = \Sigma_i a_i g_i$ of the g_i with $du(q) = df - \frac{f(q)}{\Phi(q)} d\Phi$. The function f is harmonic at q if and only if $[f - \frac{f(q)}{\Phi(q)}\Phi - u] \in H(p)$. As $\mathfrak{H}$ is locally determined, it is harmonic if and only if for every point the above holds. $\Box$

We shall need a variant of this condition. Let $G(p)$ be the space of 2-jets of harmonic functions at the point p.

Corollary 3.8. *A function f on U is in $\mathfrak{H}$ if and only if $[f] \in G(p)$ for all $p \in \mathfrak{H}$.*

Proof. Suppose $[f] \in G(p)$. Then as above we can find harmonic functions Φ and u with $[f - \frac{f(q)}{\Phi(q)}\Phi - u] \in Q(p)$. As $f - \frac{f(q)}{\Phi(q)}\Phi - u$ is harmonic, it follows that $[f - \frac{f(q)}{\Phi(q)}\Phi - u] \in H(p)$. As p is arbitrary, it follows by local determination that f is harmonic. The converse follows by the definition of $G(p)$. $\Box$

In the case of surfaces, where $\mathfrak{H}$ depends only on the conformal class of the metric, it is now easy to construct a sheaf $\mathfrak{H}$ not corresponding to a metric. Namely, start with $\mathfrak{H}$ defined by a metric g, and define a new sheaf $\mathfrak{H}'$ using the above proposition taking $H'(q) = H(q)$ and $f_1 \notin \mathfrak{H}$. Then the projective class of the bilinear forms ϕ and hence the conformal class of any metric corresponding to $\mathfrak{H}'$ coincides with g, but $\mathfrak{H}' \neq \mathfrak{H}$ as $f_1 \in \mathfrak{H}'$. In higher dimensions, we get the same result using a dimension count to account for the projective choice, as we have the freedom to choose Φ as well as the f_i. We shall study this in more detail in the next section, though rather than a dimension count we use homotheties giving a more elegant characterisation.

Regular, locally determined sheafs have a useful maximality property.

Proposition 3.9. *Suppose $\mathfrak{H}$ is a regular locally determined sheaf and $\mathfrak{H}'$ is a sheaf of functions such that $H(p) = H'(p)$ for all $p \in M$ and such that $\mathfrak{H} \subset \mathfrak{H}'$. Then $\mathfrak{H} = \mathfrak{H}'$.*

Proof. Let $f \in \mathfrak{H}'(U)$. For any point $p \in M$, by regularity of $\mathfrak{H}$ we can find functions $\Phi \in \mathfrak{H}(U)$ with $f(p) = \Phi(p)$ and $u \in \mathfrak{H}(U)$ with $u(p) = 0$ and $du(p) = df(p) - d\Phi(p)$. As $\mathfrak{H} \subset \mathfrak{H}'$, $\Phi \in \mathfrak{H}'(U)$ and $u \in \mathfrak{H}'(U)$, hence $f - \Phi - u \in \mathfrak{H}'(U)$. By definition of $H'(p)$, it follows that $[f - \Phi - u] \in H'(p) = H(p)$, and hence f is $\mathfrak{H}$-harmonic at p. As p was arbitrary and $\mathfrak{H}$ is locally determined, it follows that $f \in \mathfrak{H}(U)$. Thus $f \in \mathfrak{H}'(U)$ implies $f \in \mathfrak{H}(U)$. □

Remark 3.10. We get a locally determined regular sheaf $\mathfrak{H}$ with the same spaces $H(p)$ by considering any second order linear elliptic operator of the form with leading term $\triangle$. Such a sheaf is in general not the sheaf of $\square$-harmonic functions for any metric.

4. Conformal sheafs and conformal flatness

We now study when a regular, locally determined sheaf $\mathfrak{H}$ is conformally equivalent to a (conformally flat) Riemannian metric. For the rest of this section we refer to functions defined on an open set U which are in $\mathfrak{H}(U)$ as $\mathfrak{H}$-harmonic or simply harmonic where there is no possibility of confusion. Recall that any Riemannian metric agrees up to first order with the Euclidean metric. However the corresponding fact need not be true for $\mathfrak{H}$, for instance, if $\mathfrak{H}$ is the kernel of some second order operator as in the previous section.

Thus, we get two kinds of curvatures - the first order ones that are obstructions to being equivalent up to first order to the Euclidean case and, if these vanish the second-order ones that correspond to being flat at the point. We will define a *conformal sheaf* to be one for which the first order curvatures vanish, in which case we can define the *conformal curvatures*.

4.1. **Homotheties and flatness.**

To derive conditions for these, we need a useful characterisation of conformal flatness. A geometric condition that guarantees flatness at a point p turns out to be the existence of *homotheties*, i.e., conformal maps between open subsets of M that fix a point p and induce a scaling on T_pM. A conformally flat metric has a family of such homotheties as Euclidean space does.

Recall that in the Riemannian case the Weyl curvature is the only obstruction to flatness.

Lemma 4.1. *Suppose $f : M \to M$ is a conformal diffeomorphism fixing p such that $f_* : T_p(M) \to T_p(M)$ is the map $v \to cv$ with $c \neq 1$. Then the Weyl curvature at p is 0.*

Proof. By definition, $f^*g = u^2 g$ for some positive function u. If $W(g)$ denotes the $(4, 0)$ Weyl tensor of g and $v_i \in T_pM,\ i = 1, .., 4$, then

$$f^*W(g)_p(v_1, v_2, v_3, v_4) = W(g)_{f(p)}(f_*(v_1), f_*(v_2), f_*(v_3), f_*(v_4)) \tag{1}$$
$$= c^4 W(g)_p(v_1, v_2, v_3, v_4) \tag{2}$$

On the other hand, since f is an isometry between g and $u^2 g$,

$$f^*W(g)_p(v_1, v_2, v_3, v_4) = W(u^2 g)_p(v_1, v_2, v_3, v_4) \tag{3}$$
$$= u(p)^2 W(g)_p(v_1, v_2, v_3, v_4) \tag{4}$$

where we have used the conformal invariance of the Weyl tensor, namely, the property $W(u^2 g) = u^2 W(g)$.

Now, a simple calculation shows that $u(p) = c$ and hence we get

$$c^4 W(g)_p(v_1, v_2, v_3, v_4) = c^2 W(g)_p(v_1, v_2, v_3, v_4)$$

for all $v_i \in T_p(M)$. Hence we get $W_p = 0$. □

Corollary 4.2. *If there exist homotheties for a conformal structure obtained from a Riemannian metric g at every point in an open set $U \subset M$, then U is conformally flat.*

Proof. By the previous lemma, the Weyl curvature vanishes at every point. This implies conformal flatness. □

4.2. **Homotheties.** Instead of dealing with a single homothety, we deal with one-parameter families of homotheties or the vector field generated by such a family. So suppose that we have a one-parameter family of diffeomorphisms ϕ_t in a neighbourhood of p that fix p and with $\phi_{t*} = e^t I$ on $T_p(M)$. Let V be the vector field $V = \frac{d\phi_t}{dt}|_{t=0}$ that induces this flow. We study when this is a family of conformal maps.

First, consider the case of surfaces. Here the Laplacian is conformally invariant. Thus, we have the following simple criterion.

Lemma 4.3. *The one-parameter family consists of conformal diffeomorphisms if and only if for every harmonic function f, $V(f)$ is harmonic.*

Proof. Suppose ϕ_t consists of conformal maps, then $f \circ \phi_t$ is harmonic for each t, and hence so is $\frac{df \circ \phi_t}{dt}$. By the chain rule, $\frac{df \circ \phi_t}{dt} = V(f)$. The converse follows from $\frac{d}{dt} \triangle f \circ \phi_t = \triangle \frac{df \circ \phi_t}{dt} = \triangle V(f)$. By hypothesis, $V(f)$ is harmonic and hence $f \circ \phi_t$ is harmonic for each t. By Corollary **??**, ϕ_t is conformal. □

In the case of dimensions at least three, we need to impose one more condition on a homothety ϕ to ensure that Lemma **??** continues to hold. The reason is that a conformal class determines a sheaf only projectively.

Definition 4.4. A conformal diffeomorphism ϕ of a sheaf $\mathfrak{H}$ is a *homothety at p* if the following two conditions hold:

i) $\phi(p) = p$ and $(\phi_*)_p = cI$ for some constant c.
ii) If $\mathfrak{H}$ contains constants, then $\phi^* \mathfrak{H} = \mathfrak{H}$.

4.3. **Homotheties for surfaces.** A very illuminating example is that of $\mathbb{R}^2$ with the Euclidean metric. Recall that every surface with a Riemannian metric is conformally equivalent (locally) to this case.

In this case, let x and y be the usual co-ordinates (in general we take harmonic co-ordinates). Let $V = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$. Then as x and y are harmonic, $V(x) = f$ and $V(y) = g$ are harmonic.

Further, the functions xy and x^2-y^2 are harmonic. This implies that $\triangle V(xy) = 0$ and $\triangle V(x^2-y^2) = 0$. Using the fact that x, y, f and g are harmonic, the equations reduce to $< dy, df > + < dx, dg >= 0$ and $< dx, df > - < dy, dg >= 0$. In other words, the equations that f and g must satisfy are

$$\triangle f = 0 = \triangle g$$

$$\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$

and

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$$

The latter equations are just the Cauchy-Riemann equations which *imply* that f and g are harmonic! Thus, we have many homotheties coming from holomorphic maps. For a general $\mathfrak{H}$, however, the latter conditions are different and so *do not* imply harmonicity. As in the case of the Cauchy-Riemann equations, we can still derive a second order equation whose leading term is the Laplacian. The first and second order terms of this give us curvatures, as we see below.

4.4. **First-order curvatures.** Suppose now that $\mathfrak{H}$ is a regular, locally determined sheaf containing constants. In this section we study when $\mathfrak{H}$ is the sheaf of $\Box$-harmonic functions associated to a Riemannian metric. First consider the case of a surface. Pick harmonic co-ordinates x and y at p, i.e., harmonic functions x and y in $\mathfrak{H}$ that vanish at p and are orthonormal (by our convention this only means orthogonal with the *same* norm) at p. Then there are harmonic functions u and v at p that up to second order are of the form $u = xy$ and $v = x^2 - y^2$. We do not in general expect a conformal homothety, only a homothety conformal up to first order (since Riemannian metrics are flat up to first order), which we define precisely below.

Definition 4.5. A function f is said to be of *order at least* k at p if, in any Riemannian metric $\frac{f(q)}{d(p,q)^k}$ is bounded on a neighbourhood of p. A function f is said to be *harmonic up to order* k if $f = \phi + g$ with ϕ harmonic and g of order at least $k+1$.

We shall often simply say f is of order k if f is of order at least k.

Definition 4.6. The vector field V is said to be conformal up to first order if for f harmonic and of order k, with $k \leq 2$, $V(f)$ is harmonic up to order $k+1$.

Let $V = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ as before. If V corresponds to a homothety, as before f and g are harmonic up to second order. Furthermore, $V(u)$ and $V(v)$ are harmonic up to third order. This gives equations that up to first order are of the form

$$\frac{\partial}{\partial y}f + \frac{\partial}{\partial x}g + l_1(x,y)\frac{\partial f}{\partial x} + l_2(x,y)\frac{\partial f}{\partial y} + m_1(x,y)\frac{\partial g}{\partial x} + m_2(x,y)\frac{\partial g}{\partial y} + c_1 f + c_2 g = 0$$

and

$$\frac{\partial}{\partial x}f - \frac{\partial}{\partial y}g + l_3(x,y)\frac{\partial f}{\partial x} + l_4(x,y)\frac{\partial f}{\partial y} + m_3(x,y)\frac{\partial g}{\partial x} + m_4(x,y)\frac{\partial g}{\partial y} + c_3 f + c_4 g = 0$$

with l_i and m_i linear functions in x and y vanishing at the origin. As in the derivation that Cauchy-Riemann equations imply harmonicity, we differentiate the first equation with respect to x (respectively y), the second with respect to y (respectively x) and add. Using the fact that f and g are harmonic up to second order, and $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} = 1$ and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0$ at the origin (as V induces homotheties), we obtain *homothety equations*. These equations at the origin give homogeneous linear equations in the coefficients of l_i and m_i, and hence in the third order coefficient of u and v. We get different functions for the different choices of u and v (i.e., choices of harmonic functions with second order terms xy and $x^2 - y^2$, respectively), thus giving a family of linear equations $L(u_{ijk}, v_{ijk}) = 0$ which we can regard as a linear function on the space of 3-jets. We call $L(u_{ijk}, v_{ijk})$ the *first-order curvature.*

Definition 4.7. The sheaf $\mathfrak{H}$ is said to be *conformal* if its first-order curvature vanishes.

In the case of higher dimensions, we use the conformal Laplacian in place of the Laplacian. We obtain analogous equations for each pair of co-ordinates.

4.5. **Homogeneous co-ordinates.** Suppose now that we have a vector field V that induces a homothety at $p \in M$ conformal up to first order. We construct *homogeneous co-ordinates* up to first order for M near p with respect to V.

Definition 4.8. A function f is said to be *homogeneous* of *degree* n and *up to order* k if $V(f) = nf + g$ where g is of order at least $k+1$.

We begin by making an elementary observation.

Lemma 4.9. *Let f be a homogeneous polynomial of degree k in the harmonic co-ordinates x and y. Then $V(f) = kf + g$ where g is of order at least $k+1$.*

Proof. As $V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + O(x^2+y^2)$, the result follows by a computation (analogous to Euler's theorem for homogeneous functions). □

We can now construct homogeneous co-ordinates.

Proposition 4.10. *We can choose harmonic co-ordinates x and y such that $V(x) - x$ and $V(y) - y$ vanish up to third order.*

Proof. Recall that $V(x)$ is harmonic up to second order as V is a dilatation up to first order and x is of first order. Thus $V(x) = x + g + \phi$ where g is harmonic and of order 2 and ϕ has order 3. By the above lemma, for some h of order at least 3,

$$V(x-g) = V(x) - V(g) = (x + g + \phi) - 2g + h = x - g + \phi + h.$$

Thus, $x - g$ gives the required co-ordinate. A similar argument gives the result for y. □

Corollary 4.11. *For x and y as above, $V(xy) = 2xy + h$ and $V(x^2 - y^2) = 2(x^2 - y^2) + k$ where h and k are of order at least* 4.

Remark 4.12. It is immediate from the proof of the above proposition that the homogeneous co-ordinates are determined up to second order by their first order terms.

We conclude that harmonic functions are standard up to third order, i.e., up to third order harmonic fuctions in the homogeneous co-ordinates are the same as those in Euclidean space with the standard Cartesian co-ordinates. Recall that by hypothesis harmonic functions are standard up to order 2.

Theorem 4.13. *For some h and h' of order at least 4, $xy + h$ and $x^2 - y^2 + h'$ are harmonic.*

Proof. By regularity, for some g of order at least 3, $xy+g$ is harmonic. It follows that $V(xy+g)$ is harmonic up to third order. Now, by the above corollary and Lemma **??**, up to third order $V(xy + g) = 2xy + 3g$, and hence $g = V(xy + g) - 2(xy + g)$ is harmonic up to third order. It follows that $xy = (xy + g) - g$ is harmonic up to third order as claimed. The proof for $x^2 - y^2$ is similar. □

The results of this section immediately generalise to higher dimensions by taking a system of harmonic co-ordinates x_i and repacing them by harmonic co-ordinates that are homogeneous up to third order.

4.6. **Conformal sheafs and metrics.** Sheafs corresponding to conformal classes of Riemannian metrics can now be fully characterised.

Suppose that $\mathfrak{H}$ is a definite conformal sheaf. As seen in Section **??**, we can associate to the sheaf $\mathfrak{H}$ a bilinear form ϕ, which by definiteness can be taken to be (locally) a Riemannian metric g. Let $\mathfrak{H}'$ be the sheaf of $\Box$-harmonic functions associated to g. We deduce when $\mathfrak{H}$ coincides with $\mathfrak{H}'$. Note that the sets $H(p)$ in the two cases coincide. We can make projective choices so that constant functions are harmonic. In the case of g this amounts to locally choosing the metric so that it has zero scalar curvature.

As $\mathfrak{H}$ is conformal, at each point p we have a family of homotheties that are conformal up to first order. Consider the vector field V_p generating these. These homotheties preserve $H(p)$, and hence the conformal class of the metric, up to first order.

If g is a flat metric near p, i.e., p has a neighbourhood isometric to a ball in Euclidean space, and ϕ_t is the homothety generated by V_p that (under the identification with Euclidean space) is the standard contraction $\phi_t : x \mapsto e^{-t}x$, $x \in \mathbb{R}^n$, then $\frac{d\phi_t^*(g)}{dt}|_{t=0} = g$. In general, the metric g may not satisfy this up to first order under the homothety generated by V_p, though this is satisfied up to 0th order, i.e. at a point. However, as its conformal class is invariant, up to first order we see that $\frac{d\phi_t^*(g)}{dt}|_{t=0} = v_p g$ for some function v_p.

As only the conformal class of g is determined by $\mathfrak{H}$, we seek a metric ug in the conformal class of g that is invariant up to first order. By a computation, we see that this holds if and only if

$$V_p(u) = (v_p - 1)$$

up to first order at p, for all p. Since we also want to ensure that the sheaf corresponding to ug has constants, we need the condition that

$$\triangle_g u = 0$$

(this follows from conformal invariance of $\Box$).

The first equation at a point p is equivalent to $du(p) = dv_p(p)$ as, in harmonic coordinates with $p = (x = 0, y = 0)$, one has $V_p(x,y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ up to first order and $v_p(0) = 1$. By the Poincaré lemma, this equation has a local solution if and only if the integrability condition $d\alpha = 0$, where $\alpha(x) = dv_x(x)$, is satisfied. Further, a solution u is harmonic if and only if $\delta\alpha = 0$. We will call $(d+\delta)\alpha$ the *torsion*. This vanishes if and only if we can find a metric that is preserved up to first order by the homotheties generated by V.

Roughly speaking the torsion of a conformal sheaf is non-zero if the families of dilatations at different points are not related in a manner similar to those for a metric.

Theorem 4.14. *Let $\mathfrak{H}$ be a definite conformal sheaf. Then there is a Riemannian metric on a neighbourhood of every point in M with the kernel of the corresponding Laplacian giving the sheaf $\mathfrak{H}$ if and only if the torsion vanishes.*

Proof. Suppose the torsion vanishes. Then there exists a metric g of zero scalar curvature invariant up to first order under the dilatation V at each point. Let $\mathfrak{H}'$ be the corresponding sheaf of harmonic functions. Fix a point p and consider harmonic co-ordinates x_i and x_i' for the sheaves $\mathfrak{H}$ and $\mathfrak{H}'$ that are homogeneous under V up to first order so that $dx_i(p) = dx_i'(p)$. By remark **??** x_i and x_i' agree up to second order. As constants are in the sheaves $\mathfrak{H}$ and $\mathfrak{H}'$ and $H(p) = H'(p)$ by construction, this implies that $G(p) = G'(p)$. By local determination, it follows that $\mathfrak{H} = \mathfrak{H}'$, i.e., $\mathfrak{H}$ is a sheaf coming from a metric. □

4.7. **Second-order curvature.** For the conformal structure corresponding to a Riemannian metric the first-order curvatures vanish as any Riemannian metric is isometric up to first order to a flat metric. However, we do have curvatures that correspond to the obstruction to the metric being trivial up to second order.

Given the sheaf $\mathfrak{H}$, we can define a corresponding second-order curvature as in the case of first-order curvatures starting with homogeneous co-ordinates x and y and pick u and v harmonic with $u = xy$ and $v = x^2 - y^2$ up to fourth order. Once more we deduce Cauchy-Riemann type equations and compare them with the Laplace equation. The vanishing of the first order curvature means that these equations are trivial at the origin. Considering the first order terms, we get an equation $K(u_{ijkl}, v_{ijkl}) = 0$ in the fourth order terms. We call $K(u_{ijkl}, v_{ijkl})$ the *conformal curvature*.

Theorem 4.15. *If the conformal curvatures of a conformal sheaf $\mathfrak{H}$ vanish, then the corresponding metric is conformally flat.*

Proof. As usual we make a local choice so that constants are in $\mathfrak{H}$. By hypothesis the corresponding metric has dilatations up to second order. This implies that the Weyl curvature vanishes. As this is true for all points in a sufficiently small open set, the Riemannian metric is conformally flat. □

4.8. **Carnot-Caratheodary metrics.** In the case of a Carnot-Caratheodary metric, we can consider the kernel of the $\Box_b$ operator in place of the Laplacian. The regularity of the sheaf constructed in this manner is not automatic - if the metric is Hermitian with respect to a CR-structure then regularity is related to embeddability of the CR-structure, or equivalently whether the metric is the boundary of a Stein domain.

As in the Riemannian case, we define curvatures as the obstructions to the presence of homotheties up to first and second order. The boundary of complex hyperbolic space has homotheties and is hence flat. We shall study the CR-analogues of our curvatures in future work.

5. Piecewise-conformal structures

In this section, our goal is to associate conformal structures to piecewise-linear and more generally piecewise-Riemannian manifolds. This is based on the fact that the locus of singularities of a piecewise-linear structure has codimension at least two, and harmonicity of a function is determined by its values away from a set of co-dimension two.

A *piecewise Riemannian* n-manifold (M, g) is a smooth n-manifold with a triangulation and a smooth Riemannian metric g on $M \setminus M_{n-2}$, where the M_k denotes the k-skeleton of the triangulation. The following definition is motivated by the fact that sets of codimension 2 are polar sets.

Definition 5.1. A *PL-conformal structure* on M is the projective class of a sheaf $\mathfrak{H}$ of continuous functions satisfying the following property: there is a basis of open sets $\{U\}$ for M for which the Dirichlet problem has an unique solution, i.e, if h a continuous function on ∂U, then there exists exactly one function $f \in \mathfrak{H}(U)$ with $f = h$ on ∂U. We will refer to this as the *Dirichlet property.*

Remark 5.2. Let M be a triangulated 2-manifold and p a vertex in M. If we consider an open set U containing p which is diffeomorphic to an Euclidean ball, then the definition above implies that U has the conformal structure of the punctured disc in $\mathbb{C}$ rather than an annulus of finite non-zero modulus. This is because the Dirichlet problem does not have a unique solution in the latter case.

We now come to an important source of examples for PL-conformal structures. This is the conformal class associated to a PL-flat metric, which we define below.

5.1. PL-metrics and conformal structures.

Let M be a smooth n-manifold with a triangulation T. T gives rise to a *PL-flat metric* g on M as follows: g is, by definition, a flat Riemannian metric on $M \setminus M_{n-1}$ such that each n-simplex is isometric to the regular n-simplex in $\mathbb{R}^n$ (or, equivalently, each edge i.e., 1-simplex in T has length 1).

Remark 5.3. A PL-flat metric is smooth at any point p in $M_{n-1} \setminus M_{n-2}$. This is because any small metric ball $B(p, \epsilon)$ with centre p is isometric to an Euclidean ball of radius ϵ.

In order to define harmonic functions on open subsets of M, we observe that in the case of a smooth Riemannian manifold (M, g) with a triangulation, harmonic functions are determined away from the codimension-2 skeleton M_{n-2}. More precisely, we have the following proposition.

Proposition 5.4. *Let (M, g) be a smooth Riemannian manifold with a triangulation and let f be a continuous function on M which is harmonic on the open set $M \setminus M_{n-2}$. Then f has an unique extension to a harmonic function on M.*

Proof. Let p be a point in M_{n-2} and $B = B(p, \epsilon)$ be the ball of radius ϵ and centre p and N_δ the tubular neighbourhood of M_{n-2} of radius δ. Choose ϵ small and $\delta < \epsilon$. Let $U_\delta = B \bigcap N_\delta^c$. We can write $\partial U_\delta = H_\delta \bigcup G_\delta$, where $H_\delta \subset \partial B$ and $G_\delta \subset \partial N_\delta$.

Let h be the solution to the Dirichlet problem on B for $\triangle$ with boundary values given by f. Then $h - f$ is the solution to the Dirichlet problem in U_δ with boundary values given by $h - f$. For any fixed q in $B \setminus N_\delta$, we have

$$(h-f)(q) = \int_{H_\delta} (h-f)P + \int_{G_\delta} (h-f)P,$$

where P denotes the Poisson kernel.

The first integral is zero since $h = f$ on H_δ. As $\delta \to 0$, we claim that the second integral goes to zero. Since $h - f$ is bounded, it is enough to show that $\int_{G_\delta} P$ goes to zero.

We use the fact that M_{n-2} is a *polar set* as it has codimension two. That M_{n-2} is polar follows by a straightforward extension of results from classical potential theory (see for instance [**?**] for the Euclidean case). Recall that by definition of a polar set this means we can find a subharmonic function ϕ on B such that $\phi(x) \to \infty$ as x approaches $B \bigcap M_{n-2}$. Fix $x \in B \setminus \{p\}$.

Note that for δ sufficiently small, $x \in U_\delta$. As ϕ is subharmonic, we have

$$\phi(x) \geq \int_{H_\delta} \phi P + \int_{G_\delta} \phi P,$$

As $\delta \to 0$, $\phi|_{G_\delta} \to \infty$. Hence $\int_{G_\delta} P$ as claimed.

Thus h is harmonic and $h(q) = f(q)$ for any $q \in B \setminus M_{n-2}$, i.e., h is a harmonic extension of f. □

The above proposition justifies the following defintion.

Definition 5.5. A bounded function defined on an open set in M is *harmonic* if it is harmonic on the complement of the codimension-2 skeleton. The *conformal class* of the PL metric on M is the projective class of the sheaf $\mathfrak{H}$ of harmonic functions on M.

We have the following easy proposition, which verifies that conformal classes of PL-flat metrics are indeed PL-conformal structures.

Proposition 5.6. *Let $\mathfrak{H}$ be the sheaf of harmonic functions associated to a PL-flat metric g. Then $\mathfrak{H}$ satisfies the Dirichlet property and is conformally flat away from M_{n-2}. Hence $\mathfrak{H}$ defines a PL-flat conformal structure.*

Proof. The argument for both existence and uniqueness for the Dirichlet problem on an open set U with boundary values given by f are as in Proposition **??** as M_{n-2} is polar. □

Note that by the above propositions, the PL-conformal structure associated to a PL-metric, or a piecewise-Riemannian metric, is well-defined, i.e., if we consider the PL-metric as associated to a subdivision of the original triangulation, we get the same sheaf of harmonic functions. Furthermore if we view a Riemannian metric as a piecewise-Riemannian metric with respect to some triangulation, then the sheaf corresponding to the associated piecewise-conformal structure is the same as that associated to the Riemannian metric.

Thus, we have a generalisation of the usual notion of a conformal structure which is consistent with the classical case.

6. Convergence and degeneration

We now study convergence and degeneration of conformal structures. We begin by defining pointwise convergence.

Definition 6.1. A sequence of conformal structures given by sheafs $\mathfrak{H}_i$ converges at p if the corresponding vector spaces $H_i(p) \subset Q(p)$ converge to a space $H(p) \subset Q(p)$, which we call the limit at p.

Such a pointwise limit may be degenerate, even if the limit exists at each point. We shall define when such a limit is regular. First, we define the limit of a sequence of sheafs of functions.

Definition 6.2. Let $\mathfrak{H}_i$ be a sequence of sheafs of continuous functions. Then we define the limit sheaf $\mathfrak{H} = lim \mathfrak{H}_i$ by

$$\mathfrak{H}(U) = \{f \in C(U) : f = lim_{i\to\infty} f_i, f_i \in \mathfrak{H}(U)\}$$

where the convergence of functions is uniform convergence on compact sets.

Remark 6.3. Since the limit of harmonic functions is harmonic, the above definition gives the correct sheaf when we actually have convergence of Riemannian metrics.

In dimension 2 we immediately have a notion of conformal structures converging since the sheafs $\mathfrak{H}_i$ are well defined. In higher dimensions, the $\mathfrak{H}_i$ are defined only projectively, so we make the following definition.

Definition 6.4. The sequence of projective classes of sheafs $[\mathfrak{H}_i]$ converges regularly to $\mathfrak{H}$ at p if there is a neighbourhood U of p such that there is a choice of sheafs $u_i \mathfrak{H}_i$ restricted to U which converge to a sheaf $\mathfrak{H}$ on U so that p is a regular point of $\mathfrak{H}$. We say the the convergence is regular *on* U if it is regular at every point of U.

As we want to allow the case of piecewise-conformal structures, we weaken the above notion of regular convergence. Recall that a neighbourhood of a regular point consists of almost regular points.

Definition 6.5. The sequence of projective classes of sheafs $[\mathfrak{H}_i]$ converges weakly-regularly to $\mathfrak{H}$ on an open set U if there is a choice of sheafs $u_i \mathfrak{H}_i$ restricted to U which converge to a sheaf $\mathfrak{H}$ on U so that q is a regular point of $\mathfrak{H}$ for every point in a subset $U - K$ with K having codimension at least 2 in U.

Remark 6.6. While one needs to make a projective choice here, this is only a pointwise choice rather than uniform control over the curvature tensor. Furthermore regularity is a very mild condition that needs to be satisfied.

We now show that the limiting sheaf is well-defined when we have a sequence of Riemannian metrics $\{g_i\}$ converging to a metric $\{g\}$, i.e., we show that the limiting sheaf is that determined by $\{g\}$ even if the metrics $\{g_i\}$ are changed conformally. More precisely,

Proposition 6.7. *Suppose that $g_i \to g$ smoothly. Let $\mathfrak{H}_i$ and $\mathfrak{H}$ be the sheafs of $\Box$-harmonic functions associated to g_i and g respectively. If $v_i \mathfrak{H}_i$ converges regularly to a locally determined sheaf $\mathfrak{H}'$, then given any point p there is an open set U containing p and a function v such that $\mathfrak{H}'(U) = v\mathfrak{H}(U)$.*

Proof. We begin by scaling the metrics g_i and g locally to $u_i g_i$ and ug such that they have zero scalar curvature so that $\triangle = \Box$. We need to ensure that this can be done in such a manner that $u_i g_i \to ug$.

Firstly, we pick u defined on a ball B with centre p such that ug has zero scalar curvature and replace g_i by ug_i. Now we have g_i converging to a metric of zero scalar curvature and we need to find u_i such that $u_i \to 1$ (C^2 convergence). We shall construct the u_i by solving a Dirichlet problem. The proof is similar to the latter part of the proof of Lemma **??**

Recall that u_i are solutions of $\Box_i u = 0$, where $\Box_i := \Box_{g_i}$ (similarly for $\triangle_i$ and s_i). We take a fixed ball B around a point p and consider the Dirichlet problem $\Box_i u_i = 0$ on B and $u_i|_{\partial B} = 1$. The Schauder estimates of Theorem **??** for $\triangle_i$ applied to $u_i - 1$ give

$$\|u_i - 1\|_{C^{2,\alpha},B} \leq C_i \|s_i u_i\|_{C^\alpha,B}.$$

Since $g_i \to g$ it is seen that C_i remains bounded as $i \to \infty$. To bound u_i, we again use the *a priori* estimate of Theorem **??** applied to the differential operator $\Box_i$. This can be done if the ball B is sufficiently small, which we assume is the case since our result is local.

$$sup_B|u_i| \leq E_i \; sup_{\partial B}|u_i|.$$

As before E_i remains bounded as $i \to \infty$. Since $u_i|_{\partial B} = 1$ and since $s_i \to 0$, we see that $\|u_i - 1\|_{C^{2,\alpha},B} \to 0$.

Now we return to the main proof.

Let $\tilde{g}_i = u_i g_i$, $\tilde{g} = ug$ and $\tilde{\mathfrak{H}}_i$, $\tilde{\mathfrak{H}}$ the respective sheafs. Then the hypothesis is that $w_i \tilde{\mathfrak{H}}_i \to \mathfrak{H}'$, where $w_i = v_i u_i^{\frac{2-n}{4}}$.

Note that all the sheafs involved are invariant under scalar multiplication. Hence, by scaling w_i by $inf w_i$, we can assume that $w_i(x) \geq 1$ for any $x \in B$ and $inf_x w_i(x) = 1 \; \forall i$. Now by regularity of $\mathfrak{H}'$ there exists $f \in \mathfrak{H}'(B)$ with f positive and bounded. Choose $f_i \in \tilde{\mathfrak{H}}_i(B)$ with $lim_{i\to\infty} w_i f_i = f$ and $f_i > 0$. Since $w_i \geq 1$, we see that the f_i are bounded above and below (by 0).

Thus f_i are a sequence of functions bounded above and below that are harmonic with respect to the metrics $\tilde{g}_i$. As $\tilde{g}_i \to \tilde{g}$, we get uniform estimates for the Harnack inequality. Hence, there is a convergent subsequence $f_{i_j} \to h$, where $h \in \mathfrak{H}(B)$. Further, as $inf_x w_i(x) = 1 \; \forall i$, we deduce that $h \neq 0$ and hence, as h is harmonic and non-negative, $h(x) > 0 \; \forall x$. From this it follows that $w_{i_j} \to w$ smoothly, for some smooth function w.

Hence $w_{i_j} \tilde{\mathfrak{H}}_{i_j} \to w\tilde{\mathfrak{H}}$. We now show that $w_i \tilde{\mathfrak{H}}_i \to w\tilde{\mathfrak{H}}$. By the definition of limits, $\mathfrak{H}' \subset w\tilde{\mathfrak{H}}$. But as both $\mathfrak{H}'$ and $w\tilde{\mathfrak{H}}$ are regular and locally determined (as such sheaves are maximal by Proposition **??**) we get $\mathfrak{H}' = w\tilde{\mathfrak{H}}$.

Since $\tilde{\mathfrak{H}} = u^{\frac{2-n}{4}} \mathfrak{H}$ we conclude that $\mathfrak{H}' = wu^{\frac{2-n}{4}} \mathfrak{H}$ and hence we can take $v = wu^{\frac{2-n}{4}}$.

□

The limit of conformal structures is in general not one corresponding to a Riemannian metric even pointwise. However, we do get a (projective class of) sub-Riemannian metrics associated to the limit.

Theorem 6.8. *Suppose a sequence of conformal structures $\mathfrak{H}_i$ associated to Riemannian metrics converges to $\mathfrak{H}$ at the point p. Then we have an associated Carnot structure at p canonical up to scaling.*

Proof. At each $p \in M$, we can associate to $\mathfrak{H}$ a (projective class of) bilinear form ϕ on $T_p^*(M)$. This is non-negative definite as the forms associated to each $\mathfrak{H}_i$ are positive definite. We can define the dual norm on T_pM as

$$\|v\| = sup_{\xi \in T^*(M)} \frac{\xi(v)}{\sqrt{\phi\langle \xi, \xi \rangle}}$$

where we make the convention that $a/0 = \infty$ for $a > 0$.

This is a norm on T_pM except that it takes the value ∞ outside a subspace of T_pM. Thus, we get a Carnot structure at p. □

If the conformal structure is almost regular at p, then we may instead have a norm which is ∞ everywhere, giving a maximally degenerate Carnot structure.

7. Concluding remarks

In the previous section, we studied convergence and degeneration of conformal structures on a *fixed* background manifold M. This is analogous to studying limits of *Riemannian metrics* g, rather than *Riemannian manifolds*. Thus, this should be viewed as merely setting a framework for studying the more interesting question of convergence of manifolds equipped with conformal structure.

Well known instances of convergence of conformal structures are the Deligne-Mumford compactification of Moduli space and Thurston's compactification of Teichmüller space. It would be very useful to have, at least in special cases, an analogue of these in higher dimensions.

One possible application of such a compactification may be the study of the Yamabe invariant (also referred to as the *Sigma constant* by some authors) of a manifold cf. [**?**]. One of the difficulties of calculating this number is that a sequence of Yamabe metrics (i.e., Riemannian metrics optimising the total scalar curvature functional within conformal classes) need not converge to a Riemannian metric. An alternative approach is to consider a sequence of *conformal classes* whose Yamabe constants converge to the Yamabe invariant. If these converge to a conformal class of a Riemannian metric, we are done. Else we extend notions of Yamabe constants to the case of degenerate conformal structures and study where the supremum on the compactification is attained.

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E-mail address: gadgil@isibang.ac.in

E-mail address: harish@isibang.ac.in

Stat-Math Unit,, Indian Statistical Institute,, Bangalore, India