A TRIANGULATION OF A HOMOTOPY-DELIGNE-MUMFORD COMPACTIFICATION OF THE MODULI OF CURVES

SIDDHARTHA GADGIL

Abstract. We construct a triangulation of a compactification of the Moduli space of a surface with at least one puncture that is closely related to the Deligne-Mumford compactification. Specifically, there is a surjective map from the compactification we construct to the Deligne-Mumford compactification so that the inverse image of each point is contractible. In particular our compactification is homotopy equivalent to the Deligne-Mumford compactification.

1. Arc systems and Cells

We construct a compactification of the Moduli space of a surface with at least one puncture which is homotopy equivalent to the Deligne-Mumford compactification of moduli space. Thus, the (co)homology of the Deligne-Mumford compactification can be computed using our traingulated compactification.

It is easy to see that our methods extend to the case of surfaces with boundaries and punctures (including boundary-punctures). This work is motivated in part by an attempt to obtain a combinatorial description of the Heegaard Floer theory of Ozsvath and Szabo [9][10]. The boundary map for Heegaard Floer theory can be defined in terms of counting pairs of maps [6]. One expects that this can be expressed in terms of cup products on spaces related to Deligne-Mumford compactifications of Moduli spaces. Further, Lefschetz pencils on 4-manifolds can also be described in terms of the Deligne-Mumford compactification.

Our starting point is the well known model of the Moduli space in terms of arc systems due to Harer [4]. We begin by recalling this model. We shall use the version of this from hyperbolic geometry due to Bowditch and Epstein [2] (see also [11]). We remark that Penner and McShane [12] have a more direct approach to the Deligne-Mumford compactification. However, our approach gives a simpler description of a compactification of the desired homotopy type.

Let F be a fixed punctured surface of finite type. We shall also regard F as a surface with boundary in the natural way. An *arc system* α is a collection of disjoint, essential, pairwise non-isotopic arcs. Throughout we shall regard isotopic arcs as equal. By splitting a surface F along a proper codimension-one manifold α we mean taking the completion of $F - \alpha$ with respect to the restriction of a complete Riemannian metric on F.

Definition 1.1. An arc system α is said to be proper if the each component of F split along α is either a disc or an annulus A with exactly one boundary component contained in ∂F.

Date: June 14, 2010.

1991 *Mathematics Subject Classification.* 57M99; 32G15.

A *weighted* arc system ξ is an arc system α together with positive weights associated to the arcs in α. We regards weighted arc systems with proportional weights as equal. We call α the support of the weighted arc system ξ.

A model $X(F)$ for the product $\mathcal{M}(F) \times \Delta$ of the moduli space of F with the simplex Δ with vertices boundary components of F is given by weighted arc systems (up to homeomorphism) with support a proper arc system. This is naturally a subset of the simplicial complex formed by all weighted arc systems, and inherits the topology of this complex. We get a model of $\mathcal{M}(F)$ by fixing a boundary component B of F and considering weighted arc system with proper support disjoint from all boundary components other than B.

1.1. **Size relations.** Observe that for the boundary components in ∂F that are contained in an annular component of F split along α are exactly those that are disjoint from α. We shall call such boundary components *small* with respect to α and other boundary components *large*.

Remark 1.2. If α is a proper arc system, then two small boundary components are in different components of F split along α.

Remark 1.3. A curve system α is proper if and only if every essential simple closed curve in F that is disjoint from α is isotopic to a boundary component, which is necessarily small.

The compactification is described by arcs systems on surfaces obtained from splitting F. These satisfy compatibility condiditions, given in terms of what we call *size relations*.

Definition 1.4. A *size relation* on a (finite) set S is an equivalence relation $\sim$ on S together with a partial order $\ll$ on the set of equivalence classes.

We shall also regard $\ll$ as a partial order on the set S which is compatible with the given equivalence relation.

Definition 1.5. An element $s \in S$ is said to be *small* if there is an element $s' \in S$ such that $s \ll s'$. Otherwise we say s is *large*.

Observe that if S is finite then there are large elements. Consider a size relation $\ll$ on the set of boundary components of F.

Definition 1.6. A proper arc system α is said to be compatible with $\ll$ if the following conditions hold.

- If C and C' are boundary components of F that intersect α, then $C \sim C'$.
- If a boundary component C of F is disjoint from α and another component C' intersects α, then $C \ll C'$.

Note that any proper arc system α on F is compatible with some size relation. Namely, we define $C \sim C'$ if and only if both C and C' intersect α or $C = C'$, and $C \ll C'$ if and only if C is disjoint from α and C' intersects α. We call this size relation the *minimal* size relation for α.

1.2. **Cells in the strata.** A proper arc system α in F corresponds to a cell with dimension $|\alpha| - 1$, with $|\alpha|$ the number of arcs in α (the dimension is reduced by one due to projectivisation). We shall consider additional cells in various strata of our compactification. As with the Deligne-Mumford compactification, these strata correspond to *curve systems* $\mathcal{C}$, i.e., collections of disjoint, essential, pairwise non-isotopic curves.

Choose and fix a size relation $(\sim, \ll)$ on the components of ∂F. Let $\mathcal{C}$ be a curve system and let $\widetilde{\mathcal{C}} = \mathcal{C} \cup \partial F$. Consider a size relation $(\sim, \ll)$ on the components of $\widetilde{\mathcal{C}}$ (which we simply call a size relation on $\widetilde{\mathcal{C}}$) extending the given relation on ∂F.

Let $F_{\mathcal{C}}$ be the surface obtained by splitting F along $\mathcal{C}$. We say that two components $C, C' \subset \widetilde{\mathcal{C}}$ are *adjacent* if they are both contained in the closure of some component of $F - \mathcal{C}$. Equivalently, C and C' are both contained in the boundary of a component of $F_{\mathcal{C}}$.

Definition 1.7. The size relation $(\sim, \ll)$ on $\widetilde{\mathcal{C}}$ is said to be *permissible* if for every $C \subset \mathcal{C}$, there is a component $C' \subset \widetilde{\mathcal{C}}$ adjacent to C such that $C \ll C'$.

Fix a permissible relation $\ll$ on $\widetilde{\mathcal{C}}$. For each component G of $F_{\mathcal{C}}$, the size relation $\ll$ restricts to a size relation on the boundary components of G.

Definition 1.8. A collection of arc systems $\gamma(G)$, G a component of F split along $\mathcal{C}$, is said to be proper if, for each component G of $F_{\mathcal{C}}$, $\gamma(G)$ is a proper arc system which is compatible with the restriction of the size relation $(\sim, \ll)$ to ∂G.

Observe that if $\mathcal{C}$ is empty this is just the set of proper arc systems on F compatible with $(\sim, \ll)$. In general, we associate a cell to a curve system $\mathcal{C}$ and a collection of arc systems $\gamma(\cdot)$, so that the collection $\gamma(\cdot)$ is proper with respect to $\mathcal{C}$ and some size relation $\ll$. The points in the cell correspond to weighted arc systems in components G of $F_{\mathcal{C}}$ with support $\gamma(G)$ considered up to scaling (separately in each component G). We shall call this space, which is a product of simplices, $\Delta(\gamma(\cdot), \mathcal{C})$. We denote the union of the products of simplices in the strata corresponding to $\mathcal{C}$ by $X(\mathcal{C})$.

Note that if we are given a proper arc system $\gamma(G)$ on each component G of $F_{\mathcal{C}}$, we can consider the minimal size relation on the components of ∂G from $\gamma(G)$. We can extend the relations $\sim$ and $\ll$ to $\widetilde{\mathcal{C}}$ by requiring transitivity. However, this may give the relation $a \ll a$, and thus we do not get a partial order on $\widetilde{\mathcal{C}}$. It is easy to see that a collection of arc systems is compatible with respect to some partial order if and only if, for the transitive relations $\sim$ and $\ll$ generated by the minimal size relations, we do not have a relation of the form $a \ll a$.

Thus, $X(\mathcal{C})$ is the subset of the product of the spaces $X(G)$ given by the condition that the relations $\sim$ and $\ll$ generated by the minimal size relations on each component give a size relation $(\sim, \ll)$ on $\widetilde{\mathcal{C}}$. The space $X(\mathcal{C})$ inherits its topology from this product of spaces.

2. Gluing arc systems

Consider a curve system $\mathcal{C}$ together with a permissible size relation $\ll$ and let $F_{\mathcal{C}}$ be the result of spliting F along $\mathcal{C}$. We shall relate proper arc systems on F with proper arc systems on the components G of $F_{\mathcal{C}}$. We say that an arc $\gamma^j \subset \gamma(G)$ in G is infinitesimal if exactly one of its boundary points lies on a small component of ∂G.

Consider an arc system α on F. Assume that α intersects $\mathcal{C}$ minimally and transversally. It is well known that in this case $\alpha \cap G$ is well defined up to isotopy for each component G.

Definition 2.1. The *restriction* of α to a component G is the arc system $\gamma(G) = res_G(\alpha)$ consisting of the arcs in the completion of $G \cap \alpha$ with both end points on large components, with isotopic arcs identified.

Lemma 2.2. *Suppose α is an arc system whose restriction to each component F is proper and so that α intersects each component of $\mathcal{C}$. Then α is proper.*

Proof. By Remark 1.3, it suffices to show that an essential curve η that is disjoint from α is homotopic to a small boundary component of ∂F. Let η be an essential simple closed curve in F. Assume γ intersects $\mathcal{C}$ minimally.

We first show that η is disjoint from $\mathcal{C}$. Suppose not, of the components of $\mathcal{C}$ that η intersects, let C be a component that is maximal with respect to the given size relation. By Definition 1.7, there is a component C' adjacent to C with $C \ll C'$. Let G be the component of $F_{\mathcal{C}}$ containing C and C' and let η' be a component of (the completion of) $\eta \cap G$.

By maximality of C, the other endpoint of η' is also contained in a component C'' of $\mathcal{C}$ so that $C'' \ll C'$. Hence both C and C'' are small, which contradicts Remark 1.2 as the restriction $\gamma(G)$ of α to G is proper.

It follows that η is isotopic to a small boundary component C of ∂G. As α intersects each component of $\mathcal{C}$, C is in contained in ∂F and is a small boundary component of F.

□

We next see that any collection of proper arc systems $\gamma(G)$ on components G of $F_{\mathcal{C}}$ is the restriction of a proper arc system α. Furthermore, the extension of an arc is, in an appropriate sense, at least as large as the given arc.

Lemma 2.3. *Let $\gamma(G)$, G a component of $F_{\mathcal{C}}$, be a collection of proper arc systems. Then there is a proper arc system α on F so that the following hold.*

1. *The restriction of α to G is $\gamma(G)$.*
2. *Each component of α intersects a unique component G_α in an arc γ_α in $\gamma(G_\alpha)$ and intersects all other components in infinitesimal arcs.*
3. *If the end points of γ_α lie on components C_1 and C_{-1} of $\widetilde{\mathcal{C}}$, and C' is another component of $\widetilde{\mathcal{C}}$ that α intersects, then $C_1 \sim C_{-1}$ and $C_1 \ll C'$.*
4. *Each component of α_i is contained in a unique component of α.*

Proof. We shall extend each arc $\gamma^j \subset \gamma(G)$ to a proper arc α^k in F by attaching infinitesimal arcs disjoint from all the other arcs of the collections $\gamma(G') \subset G'$. By iterating this procedure, we obtain the system α.

Let C_1 and C_{-1} be the curves in $\partial G \subset \mathcal{C}$ on which the end points of γ^j lie. By definition of a proper system, both these curves are large in G, in particular $C_1 \sim C_{-1}$. Hence, as $\ll$ is permissible, if C_1 is not in ∂F, then C_1 is small in the other component G_1 in which it is contained. It follows that C_1 is the boundary of an annulus A in the surface obtained from G_1 by splitting along $\gamma(G_1)$. Let C_2 be a component of ∂G_1 that intersects A. Extend γ^j by an infinitesimal arc from C_1 to C_2, which can be assumed to be disjoint from any given collection of infinitesimal arcs if needed (choosing C_2 appropriately). We temporarily denote this extension α^k.

Observe that C_2 is large in G_1 and hence $C_1 \ll C_2$. Thus, by permissibility of $\ll$, if C_2 is not in ∂F, C_2 is small in the other component in which it is contained. Iterating the above construction, we get a sequence of components $C_1 \ll C_2 \ll$

$C_3 \ll \dots$ and extensions of α^k of γ^j by infinitesimal arcs. As $\widetilde{\mathcal{C}}$ has finitely many components, this process must terminate with some $C_j \subset \partial F$ and an extension of γ^j to an arc α^k with an end point in ∂F. The same procedure applied to C_{-1} gives an extension of γ^j to a proper arc α^j in F.

Applying this procedure to each arc in each arc system $\gamma(G)$ in succession, and noting that this can be done keeping the new arcs disjoint, we get an arc system α whose restriction to each component G is $\gamma(G)$. This is proper by Lemma 2.2.

The rest of the statements are evident from the construction.

□

The condition (4) in Lemma 2.3 is purely for notational convenience. On the other hand, the above proof shows that conditions (2) and (3) are automatically satisfied if α restricts to proper curve systems on each component G.

3. Inclusion maps

We now describe when one cell is contained in the closure of the other and the associated topology. First, we recall the case of cells when $\mathcal{C}$ is empty.

In this case, simplices are associated to proper arc systems. Consider two proper arc systems α and α' and the associated cells $\Delta(\alpha)$ and $\Delta(\alpha')$. Then $\Delta(\alpha)$ is contained in the closure of $\Delta(\alpha')$ if and only if $\alpha \subset \alpha'$. A weighed arc system in $\Delta(\alpha)$ can be regarded as a weighted arc system corresponding to α' with weights 0 for the curves in $\alpha' - \alpha$. This gives a natural topology on $\Delta(\alpha) \cup \Delta(\alpha')$.

Next, we consider inclusions of a cell $\Delta(\gamma(\cdot), \mathcal{C})$ in the stratum corresponding to $\mathcal{C}$ in the cell $\Delta(\beta)$ corresponding to a proper arc system β in F. Assume β intersects $\mathcal{C}$ minimally. For each component G of F split along $\mathcal{C}$, let $\beta(G) = \beta \cap G$. The cell $\Delta(\gamma(\cdot), \mathcal{C})$ is in the closure of $\Delta(\beta)$ if $\gamma(G) \subset \beta(G)$ for all components G. Note that, for each component G, as $\gamma(G)$ is proper, $\beta(G)$ is a union of infinitesimal arcs and an arc system in G that is proper with respect to the minimal order from $\gamma(\cdot)$.

Note that any weighted arc system ξ with support β gives, for each component G, a weighted arc system $\zeta(G) = Res_G(\xi)$ on G by associating to an arc $\beta^j(G) \in \beta(G)$ the sum of the coefficients of arcs in β whose intersection with G is $\beta^j(G)$. A weighted arc system on G with support $\gamma(G)$ can be regarded as a weighted arc system on $\beta(G)$ with weights 0 for arcs not in $\beta(G)$. In this manner we obtain a natural topology on $\Delta(\beta) \cup \Delta(\gamma(\cdot), \mathcal{C})$.

Finally, consider two cells $\Delta_1 = \Delta(\gamma_1(\cdot), \mathcal{C}_1)$ and $\Delta_2 = \Delta(\gamma_2(\cdot), \mathcal{C}_1)$. For Δ_1 to be contained in Δ_2, we require that $\mathcal{C}_1 \supset \mathcal{C}_2$. We can then regard the cell Δ_1 as corresponding to cells in the components G of F split along $\mathcal{C}_2$. We are thus reduced to the previous case.

4. Compactness

We next see that the union $\bar{X}(F) = \cup_{\mathcal{C}} X(F, \mathcal{C})$ of the strata we have constructed is compact. Note that $X(F) = X(F, \phi)$ corresponds to the empty collection. We shall often omit F from the notation if it is clear from the context.

Theorem 4.1. *The union $\bar{X} = \cup_{\mathcal{C}} X(\mathcal{C})$ of simplices in strata over all curve systems (up to homeomorphism) is compact.*

Proof. We shall prove this by induction on the complexity of the surface F. Here the complexity of a surface is the maximum number of arcs in an arc system on F.

First, consider a sequence of points $\xi_i \in X(F)$. As there are only finitely many arc systems up to homeomorphism, by passing to a subsequence we can assume that these points have support a fixed arc system α. Further, as the weights all lie in $[0,1]$ and have sum 1, by passing to a subsequence, we can assume that the weights converge to numbers $c^j \in [0,1]$ corresponding to the components α^j of α, with the sum of the numbers c^j equal to 1.

Let $\alpha_+ \subset \alpha$ be the (non-empty) set of arcs for which the coefficients have positve limit. Let F_+ be a regular neighbourhood of the union of the arcs in α_+ and the boundary components of F which intersect α_+. We let $\mathcal{C}$ be the collection of curves consisting of the boundary of components of $\partial F - F_+$ that are not annuli and central circles of components that are essential annuli (i.e., annuli that are not parallel to a boundary component of ∂F). Up to isotopy, F_+ is a collection of components of the surface obtained from F by splitting along $\mathcal{C}$. Let F_1 be the union of the components not in F_+.

By definition of F_+, $\alpha_+ \subset F_+$ and the weighted arc systems ζ_i consisting of arcs in α_+ with weights those in ξ_i have a limit which is a proper arc system in F_+. Thus, for each component in F_+ we obtain a limiting weighted arc system.

The surface F_1 has a lower complexity than F_+. We consider the restriction η_i of ξ_i to F_1, i.e., the arc system with support the intersection of the support of ξ_i with F_1 and with the coefficient of an arc in η_i the sum of the coefficients of arcs in ξ_i that contain the given arc. By induction, on passing to a subsequence (and considering weights up to scaling in each component of F_1) we obtain a limit, which in general lies in a stratum corresponding to a curve system $\mathcal{C}'$.

It is easy to see that the sequence ξ_i converges to a point in the stratum corresponding to $\mathcal{C} \cup \mathcal{C}'$, with the size relation extended so that for each component F' of F_+, the components of $\partial F' \cap \partial F_+$ are equivalent and large while the other components are small.

In general, as the number of curve systems is finite we can assume that a sequence of points is contained in a fixed stratum corresponding to a curve system $\mathcal{C}$. We then apply the above argument to each component of the surface split along $\mathcal{C}$.

□

We shall use inductive arguments as in the above theorem. To do this, it is useful to observe a lemma regarding convergence to the compactification.

Suppose a sequence of weighted arc systems $\xi_i \in X$ with support α (assumed fixed) converges to a point $\bar{\xi}$ in the stratum corresponding to a curve system $\mathcal{C}$ (with a corresponding permissible size relation). Assume that the sum of the coefficients of each of the weighted arc systems ξ_i is 1. Let α_+ be the set of arcs in α whose coefficients do not converge to 0 and let $\partial_+ F$ be the union of boundary components whose coefficients (i.e., the total coefficients of arcs on them) do not converge to 0. Assume α intersects $\mathcal{C}$ minimally. Let F_+ be the union of the components of F split along $\mathcal{C}$ that intersect $\partial_+ F$.

Lemma 4.2. *We have $\alpha_+ \subset F_+$ and α_+ is a proper arc system in F_+.*

Proof. First we claim α is disjoint from $\mathcal{C}$. Suppose not, then let $C \in \mathcal{C}$ intersect α. Then, as the size relation in permissible, C is small in some component G of F split along $\mathcal{C}$. However, C intersects an arc in α_+ whose coefficients do not converge to 0, and hence do not do so on projectivisation in G (as the total coefficients of the restriction of ξ_i to G is at most 1, the total coefficient of ξ_i). This means that $\bar{\xi}$

contains an arc γ_G in G with positive coefficient with an endpoint of γ_G on C. This contradicts the assumption that C is small in G.

Thus, α_+ is contained in a union of components of F split along $\mathcal{C}$. It is clear from the definition that these are exactly the components of F_+.

Next, if G is a component of F_+, by the definition of F_+ the total coefficients of the restriction of ξ_i to G do not converge to 0. By passing to a subsequence we can assume they converge to a positive number. Hence, the coefficients of an arc in $\alpha \cap G$ in the projective limit in G converge to 0 if and only if they converge to 0 in ξ_i (without projectivising). Hence the support of the limit in G is $\alpha_+ \cap G$, which is proper by definition of the thin strata. This completes the proof.

□

We next observe that each of the strata $X(\mathcal{C})$ is contained in the closure of the moduli space X. Thus $\bar{X}$ is genuinely a compactification of X.

Proposition 4.3. *Every point in $\bar{X}$ is the limit of points in X.*

Proof. A point $\bar{\xi} \in \bar{X}$ corresponds to a curve system $\mathcal{C}$ and a weighted arc system $\bar{\xi}(G)$ in each component G of F split along C, with support a collection of proper arc systems $\gamma(G)$ in F. We assume that the coefficients $c(\gamma^j)$ of the arcs in each surface G have sum 1.

By Lemma 2.3, we can find an arc system α restricting to the systems $\gamma(G)$ and a bijective correspondence (by statement (4) of the lemma) between the arcs α^k in α and the union of arcs γ^j in the arc systems $\gamma(G)$ given by $\gamma^j \subset \alpha^k$. Hence the given weighted arc systems give a collection of coefficients $c(\alpha^k) = \gamma^j$ associated to the components of α.

Consider the permissible size relation $(\sim, \ll)$ associated to the point $\bar{\xi}$. We associate integers $\kappa(C)$ to the components of $\widetilde{\mathcal{C}} = \mathcal{C} \cup \partial F$ so that if $C \sim C'$ then $\kappa(C) = \kappa(C')$ and if $C \ll C'$, $\kappa(C) > \kappa(C')$. Each arc α^k contains a unique arc $\gamma^j \subset \gamma(G)$ for some component G (with the other subarcs of α^k infinitesimal). The end points of γ^j lie in components C and C' that satisfy $C \sim C'$. Hence we can define $k(\alpha^k) = k(C)$. Further, as all large boundary components of a component G are equivalent, we can define $\kappa(G) = \kappa(C)$ for C a large boundary component of G. We then have $\kappa(\alpha^k) = \kappa(G)$.

By statement (3) of Lemma 2.3, if an arc α^l intersects G in an infinitesimal arc, then $\kappa(\alpha^l) > \kappa(G)$. Consider the sequence ξ_i of points in X corresponding to weighted arc systems with support α and with the coefficient of the arc α^k being $\delta^{-\kappa(\alpha^k)} c(\alpha^k)$. We claim that this converges to the point $\bar{\xi}$. For, the restriction of ξ_j to a component G is the sum of $\delta^G \bar{\xi}_i$ and terms with coefficients $O(\delta^l)$ with $l > k$. It follows that, on projectivisation, the restrictions converge to $\bar{\xi_j}$. As this holds for all components G of F split along $\mathcal{C}$, the claim follows.

□

5. The canonical map to the Deligne-Mumford compactification

Let $\mathcal{M}(F)$ denote the Moduli space of curves and $\bar{\mathcal{M}}(F)$ the Deligne-Mumford compactification. Then $\bar{\mathcal{M}}(F) = \cup_{\mathcal{C}} \mathcal{M}(\mathcal{C})(F)$ is a union of strata corresponding to curve systems $\mathcal{C}$. The stratum corresponding to $\mathcal{C}$ is the moduli space of the surface obtained from F by splitting along $\mathcal{C}$.

Thus, the map from proper weighted arc systems of a surface to the moduli space of the surface, applied to F and surfaces obtained by splitting F along curve systems gives a map $\Phi : \bar{X}(F) \to \bar{\mathcal{M}}(F)$.

We define the *weight* of a curve C with respect to a weighted arc system to be the maximum of the coefficients of arcs that intersect C essentially.

Theorem 5.1. *The map Φ is continuous.*

Proof. Consider first a sequence of points $\xi_i \in X$ converging to a point $\bar{\xi} \in \bar{X}$. As in the proof of Theorem 4.1, we can assume that ξ_i have a fixed support α and the coefficients of each arc in α converge to a non-negative real number.

As $\bar{M}$ is compact, some subsequence of $z_i = \Phi(\xi_i)$ converges to a limit $\bar{z}$. Clearly it suffices to show that for every such convergent subsequence, $\bar{z} = \Phi(\bar{\xi})$. Hence it suffices to consider the case where the sequence z_i converges to a limit $\bar{z}$.

Assume that $\bar{z}$ lies in the stratum corresponding to $\mathcal{C}$. Let $\partial_+ F$ be the set of components of ∂F whose coefficients do not converge to 0 and α_+ be the set of arcs in α whose coefficients do not converge to 0. Let F_+ be a regular neighbourhood of $\alpha_+ \cup \partial_+ F$. Recall that this can also be described in terms of the limit as in Lemma 4.2.

Let F^+ be the union of components of F split along $\mathcal{C}$ that intersect $\partial_+ F$.

Lemma 5.2. *We have $F_+ = F^+$.*

Proof. We shall use the correspondence between X and $\mathcal{M}$ using hyperbolic geometry due to Bowditch-Epstein. In their construction, the weighted arc system is determined by a spine which is the locus of the points for which the shortest arc joining the point to the union of an appropriate collection of horospheres is not unique. The total coefficient of each boundary component is fixed and the horocycles are determined by these coefficients. The weighted arc system is dual to the spine constructed and the weight of an arc is determined by the length subtended by the side dual to this arc in a horosphere.

The hyperbolic structure on the surface F^+, with the curves of in $\mathcal{C}$ represented by geodesics, converges to a cusped hyperbolic structure on F^+. It is easy to see that the spine, and hence the weighted arc system also converge to those for the limiting hyperbolic structure, with the total boundaries of the cusps assigned as 0 and those of the curves in $F^+ \cap \partial F$ the corresponding limits. By comparing total coefficients, it also follows that the coefficients of arcs not contained in F^+ vanish.

As the coefficients of arcs not contained in F^+ vanish, $\alpha_+ \subset F^+$. Further, as the total coefficients in each component of F^+ do not vanish, the support of the limiting arc system is α_+, which is hence proper. The claim follows.

□

The rest of the proof follows by induction on the complexity of the surface, using the description of F_+ in terms of the limiting curve system for ξ_i from Lemma 4.2. Namely, if $\mathcal{C}'$ is the curve system corresponding to the limit $\bar{\xi}$, we observe that the components of F split along $\mathcal{C}'$ intersecting $\partial_+ F$ are isotopic to the components of F split along $\mathcal{C}$ intersecting $\partial_+ F$. In each of these components continuity follows from the continuity of the Bowditch-Epstein construction. We then proceed by induction to complete the proof. □

6. Fibres of the canonical map

Finally, we see that the fibres of Φ are contractible.

Theorem 6.1. *For a point $y \in \bar{\mathcal{M}}$, the fibre $\Phi^{-1}(y)$ is contractible.*

Proof. The point y lies in a stratum corresponding to some curve system $\mathcal{C}$. We again proceed inductively.

Firstly, choose and fix an isotopy class of complex structures on F corresponding to the point y in moduli space. Let Z be the set of isotopy classes of weighted arc systems that correspond to the given isotopy class of complex structure on F. We first show that Z is contractible by induction on the complexity of the surface. We then deduce that the fibre $\Phi^{-1}(y)$ is contractible.

Let F_0 be a component of F split along $\mathcal{C}$ containing a component of ∂F and F_1 be the union of the other components of F split along ∂F_0. Let $\mathcal{C}_1$ consist of the curves in $\mathcal{C}$ in the interior of F_1.

The point y corresponds to a pair of points (y_0, y_1), with y_0 in the moduli space of F_0 and y_1 in the stratum of the compactification of the moduli space of F_1 corresponding to $\mathcal{C}_1$.

Let Z_1 be the set of collections of weighted arc systems on the components of F_1 split along $\mathcal{C}'$, which correspond to a permissible size relation, that map to y_1 under the corresponding canonical map. By the induction hypothesis, Z_1 is contractible.

There is a natural projection from Z to Z_1 by restricting arc systems and permissible relations. We show that each fibre of this map is a disc. As Z_1 is contractible, this shows (using, for instance, [3] or [5]), that Z is contractible.

Consider a point $z_1 \in Z_1$ and let the corresponding minimal size relation be $(\sim, \ll)$. Let $\mathcal{C}_0$ be the set of boundary components of $\partial F_0 \cap \partial F_1$ that are large in ∂F_1 (hence small in ∂F_0).

Then the fibre of z_1 in Z corresponds to collections of non-negative real numbers with sum 1 associated to the components of ∂F_0 so that the minimal size relation on F_1 extends to one on F. This means that all the boundary components in $\mathcal{C}_0$ are small in F_0. Further, if $C \subset \partial F_0$ is contained in ∂F_1 and there is a component $C' \subset \mathcal{C}_0$ with $C \ll C'$ in the minimal order generated by the arc system on F_1, then C must be small in F_0.

Let $\partial^+ F_0$ consist of the components of ∂F_0 that are not in $\mathcal{C}_0$ and are not smaller than any element of $\mathcal{C}_0$. Note that $\partial^+ F_0$ is non-empty as it contains $\partial F_0 - \partial F_1$. As each component in $\partial^+ F_0$ is either not in ∂F_1 or is small in a component of F_1, both the relations $\ll$ and $\sim$ restricted to $\partial^+ F_0$ are empty (except for reflexivity of the relation $\sim$).

Hence a weighted arc system on F_0 and the given weighted arc system on F_1 form an admissible arc system with respect to a size relation on $\mathcal{C}_0$ if and only if the components of $\mathcal{C}_0$ are all small in F_0. It follows that the fibre of the projection is the simplex spanned by the components of $\partial^+ F_0$, and is hence a disc, as claimed. By the induction hypothesis, it follows that Z is contractible.

Finally, note that $\Phi^{-1}(y)$ is the quotient of Z by the group of automorphisms of a finite group. By passing to a barycentric subdivision, $\Phi^{-1}(Y)$ is the quotient of a compact, contractible space by a simplicial action of a finite set. This has trivial fundamental group by a theorem of Armstrong [1] (as each group element has a fixed point by, for example, the Lefschetz fixed point theorem). By a theorem of Oliver [8], $\Phi^{-1}(Y)$ is also acyclic (and locally contractible). By Whitehead's theorem it follows that $\Phi^{-1}(Y)$ is contractible.

□

Corollary 6.2. *The space $X(F)$ is homotopy equivalent to the Deligne-Mumford compactification of the moduli space of F.*

7. Local homoeomorphism type

A natural question to ask is whether the compactification we construct is in fact homeomorphic to the Deligne-Mumford compactification. We show that this is not so as the space $X(F)$ is not in general an orbifold.

Let F_1 be the compact surface of genus two with two boundary components C_1 and C_2 and let F_0 be the surface of genus 0 with 3 boundary components, i.e., a pair of pants. Let F be the surface obtained by identifying two of the boundary components of F_0 with the boundary components of F_1.

Consider the stratum of the compactification of the moduli space of F corresponding to the curve system $\{C_1, C_2\}$. A point y in this stratum is determined by points y_i in the moduli spaces of the surfaces F_i for $i = 0, 1$. The moduli space of the surface F_0 is a single point, and y_0 must be this point. Let y_1 be a point in the moduli space of F_1 corresponding to a Riemann surface with trivial automorphism group.

The fibre $\Phi^{-1}(y)$ corresponds to weighted arc systems in F_0 and F_1 that correspond to the given Riemann surface structures. As the arc systems are permissible, the boundary components C_1 and C_2 are small in F_0. Hence the arc system in F_0 is the unique (up to scaling) weighted proper arc system with C_1 and C_2 small. On F_1, the arc systems corresponding to the given Riemann surface structure are determined by the weights of C_1 and C_2, defined up to scaling, i.e., by the ratio of the weights of C_1 and C_2. Let x be the (unique) point in $\Phi^{-1}(y)$ with the total weights of the curves C_1 and C_2 in F_1 equal (i.e., with ratio 1).

Proposition 7.1. *A neighbourhood of the point z in $\bar{X}(F)$ is homeomorphic to the cone on an iterated suspension of a 2-torus.*

Proof. As the complex structure on F_0 is unique, a neighbourhood U of y in the Deligne-Mumford compactification is the product of a neighbourhood V of y_1 in $\bar{\mathcal{M}}(F_1)$ with discs corresponding to length and twist parameters for the curves C_1 and C_2. A length parameters 0 correspond to points in the compactification The neighbourhood V can be chosen homeomorphic to a ball.

A neighbourhood W of z is the subset of $\Phi^{-1}(U)$ consisting of those weighted arc systems with the ratio of the weights of C_1 and C_2 in $(1/2, 2)$. Observe that W intersects X and $X(\{C_1, C_2)\})$, but not the strata $X(\{C_i\})$ for $i = 1, 2$ (for which the ratios of weights is 0 or ∞).

The set $W \cap X(F)$ is the product of V with a torus corresponding to twist parameters, an interval $(0, \epsilon)$ corresponding to the sum of the weights (i.e., the total length) of C_1 and C_2 and the interval $(1/2, 2)$ corresponding to the ratio of the weights of C_1 and C_2. The set W is obtained from $W \cap X(F)$ by letting the length parameter go to 0, with no twist parameters for length 0.

Thus, the subset of W corresponding to a fixed point in V and a fixed ratio of weights is the cone on the torus corresponding to twist parameters. It follows that W is the cone on an iterated suspension of tori. □

We remark that the existence of a cell-like homotopy equivalence between spaces that are not homeomorphic does not contradict [13] as the space X is not a manifold.

Acknowledgements. I thank Dennis Sullivan for his invaluable comments and suggestions.

References

[1] Armstrong, M. A. *On the fundamental group of an orbit space.* Proc. Cambridge Philos. Soc. **61** 1965 639–646

[2] Bowditch, B. H.; Epstein, D. B. A. *Natural triangulations associated to a surface.* Topology **27** (1988), 91–117.

[3] Dydak, J; Kozlovski, G. *A Generalization of the Vietoris-Begle Theorem.* Proceedings of the American Mathematical Society, **102**, (1988) 209-212.

[4] Harer, J. L. *The virtual cohomological dimension of the mapping class group of an orientable surface.* Invent. Math. **84** (1986), 157–176.

[5] Lacher, R. C.. *Cell-like mappings I.* Pacific J. Math. **30** (1969) 717731.

[6] Lipshitz, Robert. *A cylindrical reformulation of Heegaard Floer homology.* Geom. Topol. 10 (2006), 955–1097.

[7] Luo, Feng; Stong, Richard *Measured lamination spaces on surfaces and geometric intersection numbers.* Topology Appl. 136 (2004), no. 1-3, 205–217.

[8] Oliver R. *A proof of the Conner conjecture.* Ann. Math, **103** (1976), 637–644.

[9] Ozsvath, Peter; Szabo, Zoltan *Holomorphic discs and topological invariants for closed three-manifolds*, to appear in the Annals of Mathematics.

[10] Ozsvath, Peter; Szabo, Zoltan *Holomorphic triangles and invariants of smooth four-manifolds*, preprint

[11] Penner, R. C. *The decorated Teichmller space of punctured surfaces.* Comm. Math. Phys. **113** (1987), 299–339.

[12] Penner, R.c.; McShane, G. *Stable curves and Screens on Fatgraphs*, preprint.

[13] Siebenmann, L. C. *Approximating cellular maps by homeomorphisms.* Topology **11** (1972), 271–294.

Department of Mathematics,, Indian Institute of Science,, Bangalore 560012, India
E-mail address: gadgil@math.iisc.ernet.in