NON-ORIENTABLE THOM PONTRJAGIN CONSTRUCTIONS AND SEIFERT SURFACES

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ABSTRACT. We give a non-orientable version of a relative Thom-Pontrjagin construction using as a model a triple of spaces consisting of the solid Klein bottle, the Klein bottle and an orientation reversing curve on the Klein bottle. Using this we give a proof of a theorem of Gordon and Litherland that two non-orientable Seifert surfaces for a framed knot are isotopic after finitely many stabilisations. Our construction applies to all smooth codimension two knots in spheres.

1. INTRODUCTION

The Thom-Pontrjagin construction relates framed cobordism classes of manifolds to homotopy classes of maps to model spaces. The correspondence is by associating to a map $f : M \to X$ to a model space X the inverse image $f^{-1}(x)$ for a generic point x. The cobordism class of the inverse image is independent of the point x and is the same for two homotopic maps. The map associated to a manifold is called a Thom-Pontrjagin map.

One can also associate Thom-Pontrjagin maps to submanifolds N of a given manifold. The homotopy class of the map determines whether N bounds a manifold in M. In particular, a Thom-Pontrjagin construction has been used in [3] to show that after finitely many *stabilisations*, any two orientable Seifert surfaces of a knot $K \subset S^3$ are isotopic. Here a stabilisation of Σ is a 1-surgery about an arc γ in S^3 with interior disjoint from Σ and both endpoints in the interior of Σ (more informally, adding a tube to the Seifert surface). This result allows one to construct various invariants of knots.

In this note, we study a non-orientable Thom-Pontrjagin construction and use it to address the question of uniqueness up to stabilisation for non-orientable Seifert surfaces. Our Thom-Pontrjagin construction has as a model the triple consisting of the solid Klein bottle, the Klein bottle and an orientation reversing curve on the Klein bottle. This construction involves various novel features not present in the classical case, for instance a subtle use of the structure of the fundamental group of the Klein bottle.

We next state the theorem of Gordon and Litherland which is the main consequence of our construction. Let $K \subset S^3$ be a knot and let Σ be a non-orientable Seifert surface for Σ. Let $N = S^3 - int(\mathcal{N}(K))$ be the knot exterior, where $\mathcal{N}(\cdot)$ denotes a regular neighbourhood. We first make some observations.

The longitude $\lambda = \Sigma \cap \partial N$ is a curve isotopic to K which bounds a non-orientable surface $\Sigma \cap N$ in the complement of K. This implies that λ is trivial as an element

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of $H_1(N,\mathbb{Z}/2\mathbb{Z})$ and hence the linking number of λ and K is even. Unlike the orientable case we cannot conclude that the linking number is zero. We shall call this linking number the *framing* $\mathcal{F}(\Sigma)$ associated to the Seifert surface.

In the case of the standard embedding $\mathcal{M}$ of the Möbius band in S^3 with one twist, the boundary is an unknot U and the associated framing is ± 2, with the sign depending on whether the twist was to the left or the right. Further, if K is an unknot with a non-orientable Seifert surface Σ, we can regard K as $K\#U$ with the Seifert surface $\Sigma\#\mathcal{M}$ obtained by gluing Σ and $\mathcal{M}$ along an arc. It is easy to see that $\mathcal{F}(\Sigma\#M) = \mathcal{F}(\Sigma) \pm 2$. It follows that, given an even integer $2k$, by taking band-connected sums with Möbius bands, we can obtain from Σ a Seifert surface with framing $2k$.

Note that stabilisation does not alter the framing of the Seifert surface, and hence if two Seifert surfaces are isotopic after stabilisation, they have the same framing. We show that so long as Seifert surfaces have the same framing, after stabilisation they are isotopic.

Theorem 1.1 (Gordon-Litherland [1]). *Let $K \subset S^3$ be a knot and let Σ_1 and Σ_2 be non-orientable Seifert surfaces for K with $\mathcal{F}(\Sigma_1) = \mathcal{F}(\Sigma_2)$. Then after finitely many stabilisations, Σ_1 and Σ_2 are isotopic.*

We now outline the proof of Theorem 1.1. We consider the 4-manifold $S^3 \times [0,1]$ and regard Σ_1 and Σ_2 as surfaces in $S^3 \times \{0\}$ and $S^3 \times \{1\}$ respectively. Let F be the surface $\Sigma_1 \cup (K \times [0,1]) \cup \Sigma_2$. We construct a 3-manifold M whose boundary is F using our non-orientable version of the Thom-Pontrjagin construction. Standard Morse theory arguments (very similar to the orientable case) can now be used to complete the proof.

We note that our construction applies for a smooth embedding $S^{n-2} \hookrightarrow S^n$, $n \geq 3$. In this case, it follows that two Seifert surfaces having the same framings are related by surgeries. For simplicity of notation we will stick to the classical case.

2. THE THOM-PONTRJAGIN MODEL

Let K and Σ_i, $i = 1,2$ be as in the hypothesis of Theorem 1.1. Let $N = S^3 - int(\mathcal{N}(K))$ be the knot exterior and let $W = N \times [0,1] \subset S^3 \times [0,1]$. Note that as $\mathcal{F}(\Sigma_1) = \mathcal{F}(\Sigma_2)$, we can assume that $\partial N \cap \Sigma_1 = \partial N \cap \Sigma_2$. Let $K' = \partial N \cap \Sigma_1 = \partial N \cap \Sigma_2$. We can identify K with K' and Σ_i with $N \cap \Sigma_i$. Hence $F = \Sigma_1 \cup (K \times [0,1]) \cup \Sigma_2$ is a surface embedded in ∂W. We shall construct a 3-manifold $M \subset W$ with boundary F using a variant of the Thom-Pontrjagin construction.

Consider the triple of spaces $(W, \partial W, F)$. We cannot apply the classical Thom-Pontrjagin construction to this triple as the normal bundle of F in ∂W is not trivial. However we can make a non-orientable Thom-Pontrjagin construction. Namely, let $\mathcal{K}$ be the Klein bottle, $C \subset \mathcal{K}$ be a longitude and let $V = D^2 \tilde{\times} S^1$ be the solid Klein bottle, so that $\mathcal{K} = \partial V$. Consider the triple $(V, \mathcal{K}, C)$. We shall construct a map $\varphi\colon\ (W, \partial W, F) \to (V, \mathcal{K}, C)$ with $\varphi^{-1}(C) = F$. This is the Thom-Pontrjagin map.

We shall show that we can construct a map $\psi : W \to \mathcal{K}$ with $\varphi|_{\partial W} = \psi|_{\partial W}$. Then $M = \psi^{-1}(C)$ gives the required 3-manifold. Note that the existence of the map ψ amounts to triviality of the Thom-Pontrjagin map φ.

3. Construction of φ

We shall now construct $\varphi\colon (W, \partial W, F) \to (V, \mathcal{K}, C)$ in several steps. We first construct φ on F, then on a neighbourhood $\mathcal{N}(F)$ of F, then on ∂W and finally on W. By abuse of notation we shall sometimes denote various restrictions of φ as φ.

Observe that as $H_1(N) = \mathbb{Z}$ and $W = N \times [0,1]$, we have an identification $H_1(W) \cong \mathbb{Z} \cong H_1(C) \cong H_1(V)$. We fix such an isomorphism. Assuming K is given an orientation, the isomorphism $H_1(N) = \mathbb{Z}$ is given by taking the linking number with K. In the case of $V = D^2 \tilde{\times} S^1$, we use the natural identification $H_1(V) = H_1(S^1)$ and the standard isomorphism $H_1(S^1) \cong \mathbb{Z}$.

The map on F. As $C = S^1$ is a $K(\mathbb{Z}, 1)$, to define $\varphi\colon F \to C$ it suffices to define the homomorphism $\varphi_* : \pi_1(F) \to \pi_1(C) \cong H_1(C)$. We define this to be the composition $\pi_1(F) \to H_1(F) \to H_1(W) \to H_1(C)$ with the first homomorphism being the Hurewicz map, the second being induced by inclusion and the third the isomorphism fixed in the previous paragraph. We define φ to be the map that induces this isomorphism.

Lemma 3.1. *The map $\varphi\colon F \to C$ maps orientation reversing curves on F to orientation reversing curves on the Klein bottle.*

Proof. As $\pi_1(F)$ is generated by curves supported in Σ_1 and those supported in Σ_2, it suffices to consider such curves. Let γ be a curve in $\Sigma_1 \subset N \times \{0\}$ that is orientation reversing and identify $N \times \{0\}$ with N. Then after a perturbation γ intersects Σ_1 in an odd number of points, and hence is non-zero as an element of $H_1(N, \mathbb{Z}/2\mathbb{Z}) = H_1(W, \mathbb{Z}/2\mathbb{Z})$. It follows that γ maps to an odd element in $H_1(W)$, hence $H_1(C)$. But odd elements in $H_1(C)$ are orientation reversing curves on $\mathcal{K}$. The argument for curves in Σ_2 is similar. $\square$

The map on $\mathcal{N}(F)$. We next extend φ to a neighbourhood of F. We first introduce some notation. Let $\tilde{F}$ be the orientable 2-fold cover of F and let $\tilde{C}$ be the connected 2-fold cover of C. Let τ_F and τ_C denote the deck transformations corresponding to these covers. Define involutions α_F and α_C on $\tilde{F} \times [-1,1]$ and $\tilde{C} \times [-1,1]$) by $\alpha_F(x,y) = (\tau_F(x), -y)$ and $\alpha_C(x,y) = (\tau_C(x), -y)$, respectively. Let $F \tilde{\times} [-1,1]$ (respectively $C \tilde{\times} [-1,1]$) be the quotient of $\tilde{F} \times [-1,1]$ (respectively $\tilde{C} \times [-1,1]$) by α_F (respectively α_C).

Observe that a regular neighbourhood of F is homeomorphic to $F \tilde{\times} [-1,1]$. We fix such a neighbourhood $\mathcal{N}(F)$. On the other hand the boundary of $C \tilde{\times} [-1,1]$ can be naturally identified with $\tilde{C}$ and the quotient of $C \tilde{\times} [-1,1]$ by the action of τ_C on its boundary gives the Klein bottle $\mathcal{K}$. Let C' denote the image in $\mathcal{K}$ of the boundary of $C \tilde{\times} [-1,1]$.

As φ maps orientation reversing curves to orientation reversing curves, φ extends to a bundle map $\varphi\colon F \tilde{\times} [-1,1] \to C \tilde{\times} [-1,1]$. This in turn gives a map $\varphi\colon F \tilde{\times} [-1,1] \to \mathcal{K}$ defined on a regular neighbourhood of $\mathcal{N}(F)$ of F. Note that $\varphi^{-1}(C) = F$

The map on ∂W. We next extend φ to the rest of ∂W. Observe that by construction $\varphi(\partial \mathcal{N}(F)) \subset C'$. We shall extend φ so that the complement of $\mathcal{N}(F)$ also maps into C'. As C' is a $K(\mathbb{Z}, 1)$, to construct the map on ∂W it suffices to extend $\varphi_* : \pi_1(\partial \mathcal{N}(F)) \to \pi_1(C')$ to a homomorphism $\varphi_* : \pi_1(\partial W) \to \pi_1(C')$. By construction the map $\varphi_* : \pi_1(\partial \mathcal{N}(F)) \to \pi_1(C')$ is the composition of homomorphisms

$\pi_1(\partial\mathcal{N}(F)) \to H_1(\partial\mathcal{N}(F)) \to H_1(W) \to H_1(C') \cong \pi_1(C')$ where each of the maps is a Hurewicz homomorphism or the map induced by inclusion. This extends to the map defined by the composition $\pi_1(\partial W) \to H_1(\partial W) \to H_1(W) \to H_1(C') \cong \pi_1(C')$. Hence we can extend φ to ∂W. As the complement of $\mathcal{N}(F)$ is mapped into C', we still have the property $\varphi^{-1}(C) = F$.

The map on W. We extend φ cell-by-cell. As $\pi_k(V) = 0$ for $k \geq 2$ and the only non-vanishing relative homotopy group $\pi_k(V, \mathcal{K})$ is $\pi_2(V, \mathcal{K}) = ker(\pi_1(K) \to \pi_1(V))$, it suffices to construct a homomorphism $\pi_1(W) \to \pi_1(V)$ so that the diagram

$$\begin{tikzcd}
\pi_1(\partial W) \arrow[r] \arrow[d] & \pi_1(\mathcal{K}) \arrow[d] \\
\pi_1(W) \arrow[r] & \pi_1(V)
\end{tikzcd}$$

commutes. We define the homomorphism $\pi_1(W) \to \pi_1(V)$ as the the composition $\pi_1(W) \to H_1(W) \to \pi_1(V)$ using the identification made earlier. By construction the above diagram commutes.

We can ensure that $\varphi(int(W)) \subset int(V)$. Thus we still have the property that $\varphi^{-1}(C) = F$.

4. Construction of the map ψ

We need to construct $\psi : W \to \mathcal{K}$ with $\psi|_{\partial W} = \varphi|_{\partial W}$. Equivalently, we let $\psi|_{\partial W} = \varphi|_{\partial W}$ and extend this to $\psi : W \to \mathcal{K}$. As $\mathcal{K}$ is a $K(\pi, 1)$, it suffices to extend the homomorphism on fundamental groups $\psi_* : \pi_1(\partial W) \to \pi_1(\mathcal{K})$ to a homomorphism $\psi_* : \pi_1(W) \to \pi_1(\mathcal{K})$.

Such a homomorphism exists if and only if the image under ψ_* of the kernel of the map $\pi_1(\partial W) \to \pi_1(W)$ induced by inclusion is trivial. We shall show this by obtaining a better understanding of the map ψ_*.

We identify both $N \times \{0\}$ and $N \times \{1\}$ with N. Pick a base point x_0 in ∂N and fix $y_0 = (x_0, 0)$ as the base point in W. Let $\nu : [0,1] \to \partial W$ be the path $\nu(t) = (x_0, t)$. We associate to a loop $\gamma \in \pi_1(N)$ loops $\gamma_1 = (\gamma, 0)$ and $\gamma_2 = \nu * (\gamma, 1) * \bar{\nu}$ in $\pi_1(W, y_0)$ respectively, Then $\gamma_1\gamma_2^{-1}$ maps to the trivial element in $\pi_1(W)$. Further, by the Van Kampen theorem, the kernel of the homomorphism $\pi_1(\partial W) \to \pi_1(W)$ induced by inclusion is generated by elements of this form.

Thus, we need to show that for γ_1 and γ_2 as above, $\psi_*(\gamma_1) = \psi_*(\gamma_2)$. In terms of the identification of N with $N \times \{0\}$ and $N \times \{1\}$, this is equivalent to showing that $\psi_*((\gamma, 0)) = \psi_*((\gamma, 1))$. This in turn is equivalent to showing that $\psi_*(\gamma)$ is independent of the Seifert surface $\Sigma = \Sigma_i$ of the knot K. We shall show this by computing $\psi_*(\gamma)$ in terms of the linking number between γ and K.

Recall that the Klein bottle has the fundamental group $\langle \lambda, \mu; \lambda\mu\lambda^{-1} = \mu^{-1} \rangle$. Under the inclusion map to V, the element μ maps to the identity and λ to the generator of $\pi_1(V) \cong \mathbb{Z}$. We pick a base point z_0 on C'. We can identify λ with the standard generator of $\pi_1(C')$.

Lemma 4.1. $\psi_*(\gamma) = (\mu\lambda)^{lk(K,\gamma)}$

Proof. Let m be a meridian of K. Then m is the union of two arcs $\alpha \subset \mathcal{N}(F) = F\tilde{\times}[-1, 1]$ and β, with $\alpha(t) = (y_0, t)$, $t \in [-1, 1]$ for a point $y_0 \in F$ and the interior of β disjoint from $\mathcal{N}(F)$. Without loss of generality we can assume that $\psi(y_0) = z_0$.

By construction α maps to the meridian μ of $\mathcal{K}$ and β maps to a loop in C'. Further by construction of the map $\varphi : \partial W - int(\mathcal{N}(F)) \to C'$, it follows that, as $lk(m, K) = 1$, β maps to λ. Hence m maps to $\mu\lambda$ and the lemma holds for m.

Next. let γ be disjoint from Σ. Then by construction γ maps to a loop in C'. Further, by construction of the map $\varphi : \partial(W) - int(\mathcal{N}(F)) \to C'$, $\psi_*(\gamma) = \lambda^{lk(K,\gamma)}$. Observe that $lk(K, \gamma)$ is even as Σ is dual to the generator of $H_1(N, \mathbb{Z}/2\mathbb{Z})$ and γ is disjoint from Σ. Now, in $\pi_1(\mathcal{K})$, $(\mu\lambda)^2 = \lambda^2$, hence for $k = 2m$ even, $(\mu\lambda)^k = \lambda^k$. In particular, if γ is disjoint from Σ, $\psi_*(\gamma) = (\mu\lambda)^{lk(K,\gamma)}$.

As m and the curves γ disjoint from Σ generate $\pi_1(N)$, the lemma follows. $\square$

5. Proof of Theorem 1.1

We let $M = \psi^{-1}(C)$. This is a 3-manifold in $W \subset S^3 \times [0, 1]$ with boundary F. The rest of the proof is as in the orientable case (see, for instance, [2]). We briefly sketch the proof below.

By perturbing M if necessary, we can assume that the projection $S^3 \times [0, 1] \to [0, 1]$ restricts to a Morse function on M. We can further assume that there are no critical points of index 0 and index 3, and that all critical points of index 1 are below the critical points of index 2. Let $t \in (0, 1)$ separate critical points of index 1 from critical points of index 2.

Let $\Sigma_0 = M \cup S^3 \times \{t\}$. By standard Morse theory, Σ is obtained from each of Σ_1 and Σ_2 by finitely many stabilisations. $\square$

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