ONE-HANDLES AND CONCORDANCE KERNELS FOR SMOOTH 4-MANIFOLDS

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Abstract. We consider the question of whether simply-connected smooth 4-manifolds have handle-decompositions without 1-handles. We show that for the Mazur manifold W, there is a knot $\gamma \subset \partial W$ such that for every handle-decomposition of W, the co-core of some 1-handle intersects W. Casson has shown that every handle-decomposition of W has 1-handles. However, our result is not only stronger but is also sensitive to the smooth structure of W (unlike Casson's argument).

The main technique we introduce and use is the *concordance kernel* of a smooth, closed 4-manifold M. Namely, we consider a smoothly embedded 4-ball $B \subset M$ and define the concordance kernel of M to be the set of concordance classes of knots in ∂B that are smoothly slice in $M - int(B)$. We also consider refinements of the concordance kernel that take into account the self-intersection number of a slice disc. Using Seiberg-Witten invariants as well as their refinements due to Bauer and Furuta, we show non-triviality of these refined invariants.

1. Introduction

Any smooth, compact, simply-connected manifold of dimension at least 5 has a handle-decomposition without 1-handles. This was shown by Smale in his proof of the Poincaré conjecture in these dimensions. The corresponding statement for 3-manifolds is equivalent to the Poincaré conjecture and hence is known to be true due to the work of Perelman.

Thus, the question of whether smooth, compact, simply-connected 4-manifolds have handle-decompositions without 1-handles is important in understanding the topology of smooth 4-manifolds. In the case of closed 4-manifolds, this remains open.

In this note, we prove that in a strong relative sense, every handle-decomposition of the Mazur manifold has a 1-handle. Recall that the Mazur manifold W has Kirby diagram as given in figure 1. Let γ be an unknot forming a Hopf link with the 1-handle and unlinked with the 2-handle.

Theorem 1.1. *For every handle-decomposition of W, γ intersects the co-core of some 1-handle.*

An argument of Casson (see [9]) shows that any handle-decomposition of W has 1-handles based on results related to the Kervaire conjecture in group theory. Our theorem gives a stronger result than the one obtained by Casson. More importantly, our result is sensitive to smooth structures, which can be made precise as follows.

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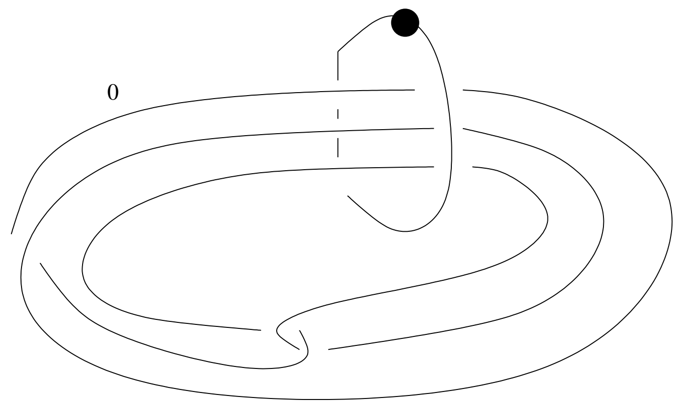

Figure 1. Mazur manifold

Note that replacing the 1-handle by a 0-framed 2-handle in the Kirby diagram gives a surgery description for $N = \partial W$. There is a symmetry interchanging the two components in this description, which induces an involution τ on N. Under the diffeomorophism τ, γ maps to a curve γ' which forms a Hopf link with the 2-handle and is unlinked from the 1-handle. This is disjoint from the co-core of the 1-handle in the given handle decomposition. The involution extends to a homeomorphism of W by a theorem of Freedman [6] but does not extend to a diffeomorphism. This has been used to deduce various exotic phenomena in [3].

The main technique we introduce and use, which may be of independent interest, is the notion of the *concordance kernel* and its refinements. We define this below and sketch some applications. These applications are essentially reformulations of known proofs and results in different terms.

Given a smooth, oriented, closed 4-manifold M, consider a smooth 4-ball $B \subset M$. Such a ball is unique up to isotopy, hence the compact manifold $\widehat{M} = M - int(B)$ is well defined up to diffeomorphism.

Observe that the boundary of $\widehat{M}$ is S^3 and the orientation determines the identification up to isotopy, where we make the convention that S^3 is oriented as the boundary of B^4 with the embedding of B^4 in M assumed to be orientation preserving. Given a knot $K \subset S^3 = \partial\widehat{M}$, we consider whether the knot is slice in $\widehat{M}$. It is easy to see that this depends only on the concordance class of K. This allows us to make the following definition.

Definition 1.2. The concordance kernel $\mathcal{C}(M) \subset \mathcal{C}$ is the subset of the classical knot concordance group $\mathcal{C}$ represented by knots $K \subset S^3$ that are slice in $\widehat{M}$.

The concordance kernel can, in principle, be used to detect exotic spheres. Further, it can be studied using the methods of Seiberg-Witten and Ozsvath-Szabo theory, which do not work directly for homotopy 4-spheres.

We have natural refinements of the invariant which we need for applications. Observe that any slice disc gives an element of $H_2(\hat{M}, S^3) = H_2(\hat{M}) = H_2(M)$, where the first identification comes from the homology long exact sequence and the second from a Mayer-Vietoris sequence. This allows one to associate to a properly embedded surface $\Sigma \subset \widehat{M}$ its self-intersection number $\Sigma \cdot \Sigma$.

Definition 1.3. For an integer $n \in \mathbb{Z}$, the concordance kernel $\mathcal{C}_n(M) \subset \mathcal{C}$ consists of classes of knots that bound a disc D in M with $D \cdot D = n$.

Using well-known methods from Seiberg-Witten theory, we show that our invariants are non-trivial.

Theorem 1.4. *There are homeomorphic manifolds M_1 and M_2 with $\mathcal{C}_1(M_1) \neq \mathcal{C}_1(M_2)$.*

The concordance kernel $\mathcal{C}_0(M)$ is more important than $\mathcal{C}_n(M)$ for general n as $\mathcal{C}_0(\cdot) = \mathcal{C}(\cdot)$ for homology 4-spheres. The usual constructions for bounding the slice genus do not work well in our case as we obtain connected sums of manifolds. However, using the invariants of Bauer and Furuta [4][5] we can deduce results about $\mathcal{C}_0(M)$.

Theorem 1.5. *Let M be the $K3$ surface and $\bar{K}$ the mirror image of the (right) trefoil knot K. Then $\bar{K} \notin \mathcal{C}_0(M)$.*

In Section 2, we use the concordance kernel as well as another notion we introduce, the *ungobbled genus*, to prove Theorem 1.1. In Section 3 we give the proofs of the rest of the applications.

2. Gobbling genus and Akbulut Corks

We now turn to the question of whether handle-decompositions of smooth, simply-connected 4-manifolds must have one-handles.

To prove Theorem 1.1, we consider the *twisted double* $M = W \coprod_{\tau:N\to N} W$, i.e., the manifold obtained by gluing together two copies of W along their boundaries using the involution τ. It was shown by Akbulut and Kirby that this is homeomorphic to S^4. However, we show that if γ is disjoint from co-cores of 1-handles, $\mathcal{C}(M) \neq \mathcal{C}(S^4)$.

The relation with 1-handles follows from whether the genus of a knot can be *localized* in a precise sense. Namely, suppose $\Sigma \subset W$ is a properly embedded surface in a smooth compact 4-manifold W. Consider smooth 4-balls $B \subset \Sigma$ such that the intersection of Σ with ∂B is connected, non-empty and transversal (hence a circle). Let $\bar{\Sigma}_B$ (the B-*gobble* of Σ) be the surface obtained from $\Sigma - (\Sigma \cap int(B))$ by attaching a disc along $\partial B \cap \Sigma$.

Definition 2.1. The *ungobbled genus* of Σ is the minimum genus of $\bar{\Sigma}_B$ over all balls B whose intersection with Σ is connected, non-empty and transversal.

Definition 2.2. Given a (possibly empty) link in $L \subset \partial W$ and a homology class in $A \in H_2(W, \partial W)$ so that L represents $\partial A \in H_1(\partial W)$, the *ungobbled genus* of A with respect to L is the minimum value of the ungobbled genus of smooth, properly embedded surfaces Σ representing A with $\partial\Sigma = L$.

In this paper, we consider W contractible and $L = K$ a knot, so A is always trivial. In this case we speak of the ungobbled genus of the knot K. However, we raise the following question for the case of closed manifolds.

Question. *Is it true that for every smooth, simply-connected 4-manifold M, the ungobbled genus of every homology class in $H_2(M)$ is zero?*

We now turn to the relative case, where W is a compact manifold with non-empty, connected boundary and K is a knot in ∂W.

Theorem 2.3. *The knot K in ∂W has ungobbled genus zero if and only if W has a proper handle-decomposition with K disjoint from the co-cores of all 1-handles.*

Proof. Observe first that for a knot $K \in \partial W$, K having ungobbled genus 0 is equivalent to the existence of an annulus A with $\partial A = K \cup K'$, with K' a knot in ∂B, where $B \subset int(W)$ is a smooth 4-ball. If such an annulus exists, then $B' = B \cup Nbd(A)$ is a 4-ball. Take B' to be the 0-handle in a handle-decomposition of W. Then K is contained in the boundary of the 0-handle, hence is disjoint from the co-cores of all 1-handles.

Conversely, as the knot K is generically disjoint from the co-cores of 2-handles and 3-handles, if K is disjoint from the co-cores of all 1-handles in some handle-decomposition, then it is isotopic to a curve on the boundary of the 0-handle. Taking B to be a ball contained in the 0-handle, it is easy to find an annulus A as above.

□

We now turn to our main application.

Proof of theorem 1.1. Consider the *twisted double* $M = W \coprod_{\tau: N \to N} W$, i.e., the manifold obtained by gluing together two copies of W along their boundaries using the involution τ. Recall [3] that γ is not smoothly slice in W while γ' is smoothly slice.

Suppose that W has a handle-decomposition with γ disjoint from the co-cores of all 1-handles. Then, by Theorem 2.3, given an embedding of the 4-ball B^4 in $W \subset M$, we can find a surface Σ with $\Sigma - (\Sigma \cap int(B^4))$ an annulus A with boundary components γ and $K \subset S^3 = \partial B^4$.

Observe that K is not slice in B^4, as if D is a slice disc for K, then $A \cup D$ is a slice disc for γ in W. Hence the mirror image $-K$ of K is also not slice, and hence K does not represent an element of $\mathcal{C}(S^4)$.

On the other hand, τ identifies γ with γ', which bounds a slice disc D in the copy of W in M not containing B. Thus, $K = \partial(A \cup D)$ is slice in $\widehat{M}$, i.e., $K \in \mathcal{C}(M)$. It follows that M is not diffeomorphic to S^4 contradicting a theorem of Akbulut and Kirby [2]. □

3. Concordance kernels and Seiberg-Witten theory

We now turn to other applications of the concordance kernel. The results in this section are essentially reformulations of known results and their proofs using concordance kernels.

We first prove Theorem 1.4.

Proof of Theorem 1.4. Observe that for a smooth manifold M, the unknot is contained in $\mathcal{C}_n(M)$ if and only if M contains a sphere with self-intersection number n. We take M_1 to be the $K3$ surface blown up at one point and M_2 to be manifold homeomorphic to M_1 which is a connected sum of 19 copies of $\mathbb{C}P^2$ and 4 copies of $-\mathbb{C}P^2$ (i.e., $\mathbb{C}P^2$ with orientation reversed). As is well known [14], M_1 does not contain a sphere of self-intersection 1 as it is an algebraic surface while M_2 obviously does. Thus the unknot is not an element of $\mathcal{C}(M_1)$ but is an element of $\mathcal{C}_1(M_2)$. □

Now we turn to the proof of Theorem 1.5. We need a variant of the standard method for finding a lower bound for the 4-genus of a knot, namely by attaching a

2-handle and embedding the result in a manifold with non-vanishing Seiberg-Witten invariants. However, in our case the manifold in which the result is embedded will be a connected sum. Thus, we need to use the refinement of Seiberg-Witten theory due to Bauer and Furuta. Specifically, the following lemma follows from a theorem of Bauer.

Lemma 3.1. *Let Q be the connected sum of two copies of the $K3$ surface. Then there is no embedded sphere S in N representing a non-trivial homology class such that $S \cdot S = 0$.*

Proof. As the canonical class of a $K3$-surface is trivial, it follows by a Theorem of Bauer [4] that for the $Spin^c$-structure s on Q with Chern class trivial, the Seiberg-Witten cohomotopy invariant is non-trivial. In particular, for any Riemannian metric on M, the corresponding Seiberg-Witten equations have a solution.

As in the proof to the adjunction inequality [12], it follows that if Σ is an embedded surface in Q with genus g with $[\Sigma] \neq 0$, then (as $c_1(s) = 0$),

$$2g - 2 \geq \Sigma \cdot \Sigma$$

In particular we cannot have a sphere in N with self-intersection zero representing a non-trivial homology class. □

We now turn to the proof of Theorem 1.5

Proof of Theorem 1.5. Let W be the manifold obtained from $\widehat{M}$ by attaching a 2-handle along a trefoil knot K contained in $S^3 = \partial\widehat{M}$ with 0-framing. As the result of attaching a 2-handle with 0-framing along the trefoil in the boundary of a ball embeds in $K3$, it follows that W embeds in $Q = K3 \# K3$.

Now suppose $\bar{K}$ bounds a disc D in M with $D \cdot D = 0$, then the union of D with the co-core of the two-handle attached to $K \subset \partial\widehat{M}$ gives a sphere $S \subset W \subset N$ representing a non-trivial element in homology. As the 2-handle was attached with 0-framing and $D \cdot D = 0$, it follows that $S \cdot S = 0$. This contradicts Lemma 3.1. □

4. Concluding remarks

The twisted double construction can be performed on many variants of the Mazur manifold, and indeed all homotopy 4-spheres are obtained in this way [7][11][13]. In each of these cases, if the result is S^4, one can show this by finitely many Kirby moves. On the other hand, if we have a curve analogous to γ but which is disjoint from the co-cores of 1-handles, this can be shown in finitely many moves. Thus the methods of this paper give a way of obtaining positive or negative evidence for the smooth Poincaré conjecture.

Another interesting question is whether $\mathcal{C}_0(M)$ in non-trivial, in the sense of distinguishing homeomorphic but not diffeomorphic manifolds. A natural class of examples to consider are the ones constructed by Fintushel and Stern that are homeomorphic to the $K3$ surface.

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