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Abstract

Recently Jammalamadaka and Mangalam (2003) introduced a general censoring scheme called the “middle-censoring” scheme in non-parametric set up. In this paper we consider this middle censoring scheme when the lifetime distribution of the items are exponentially distributed and the censoring mechanism is non-informative. In this set up, we derive the maximum likelihood estimator and study its consistency and asymptotic normality properties. We also derive the Bayes estimate of the exponential parameter under gamma prior. Since a theoretical construction of the credible interval becomes quite difficult, we propose and implement Gibbs sampling technique to construct the credible intervals. Monte Carlo simulations are performed to study the small sample behavior of the different proposed techniques. One real data set has been analyzed to illustrate the use of proposed methods.

Keywords and Phrases: Exponential distribution, Middle censoring, Consistency, Asymptotic Normality, Fixed point solution, Bayes estimate.

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1 **INTRODUCTION**

In this paper we analyze lifetime data when they are “middle-censored”. Middle censoring occurs when a data point is not observable if it falls inside a random interval. The middle censoring scheme can be thought of as a generalization of the left and right censoring scheme and clearly it is different from a double censoring scheme.

The middle censoring scheme can be described as follows. Suppose $n$ identical items are put on test and the lifetime distributions of each item are $T_1, \ldots, T_n$ respectively. For the $i$–th item, there is a random censoring interval $(L_i, R_i)$, which is distributed independent of $i$ and having some unknown bivariate distribution. For the $i$–th item, $T_i$ is observable only if $T_i \notin [L_i, R_i]$, otherwise it is not observable. Suppose $\delta_i = I(T_i \notin [L_i, R_i])$, where $I(.)$ denotes the indicator function. Therefore, when $\delta_i = 1$, the observation is not censored and we observe the actual value $T_i$, otherwise we observe only the censoring interval $[L_i, R_i]$. For the $i$–th item, we observe the following:

$$\begin{align*}
(Y_i, \delta_i) &= \begin{cases} 
(T_i, 1) & \text{if } T_i \notin [L_i, R_i] \\
([L_i, R_i], 0) & \text{otherwise.} 
\end{cases}
\end{align*}$$

(1)

Based on the observations, the problem is to estimate the lifetime distribution functions of $T_i$’s and develop necessary inferential procedures.

The middle censoring scheme was first introduced by Jammalamadaka and Mangalam [3] under the non-parametric set up. It is assumed that $T_1, \ldots, T_n$ are independent and identically distributed (i.i.d.) random variables with some unknown distribution function $F(.)$. Also, $(L_1, R_1), \ldots, (L_n, R_n)$ are i.i.d. with some unknown distribution function $G(.)$. Based on the above assumption, they obtained the non-parametric maximum likelihood estimator of the distribution function. They showed that the non-parametric maximum likelihood estimator is a *self-consistent* estimator (see the review article of Tarpey and Flury; 1996 for a nice account of the *self-consistent* estimators). But consistency of this estimator
was proved under the rather stringent condition that one of the ends is non-random.

In this paper we consider a parametric formulation of the problem. It is assumed that $T_1, \ldots, T_n$ are i.i.d. exponential random variables with mean $\frac{1}{\theta_0}$ i.e. with the probability density function (PDF) given below:

$$f(x; \theta_0) = \begin{cases} \theta_0 e^{-\theta_0 x}, & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(2)

Moreover, $(L_1, Z_1), \ldots, (L_n, Z_n)$ are i.i.d. where $L_i$ and $Z_i = R_i - L_i$ are independent exponential random variables with mean $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ respectively. It is also assumed that $\alpha$ and $\beta$ do not depend on $\theta_0$. Based on the above assumption we obtain different estimators of $\theta_0$ and study their different properties.

We provide the maximum likelihood estimator (MLE) of $\theta_0$. It is observed that the MLE can not be obtained in a closed form. We propose a simple iterative procedure for finding the MLE and the sufficient condition for the convergence of the iterative method is also provided. We also suggest the EM algorithm which can be used to compute the MLE. Sufficient condition for the convergence of the EM algorithm is also provided. It is shown that the MLE of $\theta_0$ is consistent and asymptotically normal. As might be expected, the asymptotic variance of MLE of $\theta_0$ depends on the censoring parameters $\alpha$ and $\beta$. Thus for constructing asymptotic confidence intervals for $\theta_0$ we propose to use the empirical Fisher information matrix.

We also compute the Bayes estimate of $\theta_0$ under the assumption of Gamma prior distribution on $\theta_0$. No prior distributions on the censoring parameters are assumed. Moreover, the censoring is assumed to be non-informative. It is observed that it is very difficult to compute the exact Bayes estimate in this case and we propose to use the Gibbs sampling procedure to compute the Bayes estimate and the also the highest posterior density credible interval of $\theta_0$. 

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The rest of the paper is organized as follows. In Section 2, we provide the MLE and the proposed EM algorithm. The theoretical results are provided in Section 3. We provide a Bayesian formulation of the problem in Section 4. Simulation results are presented in Section 5. For illustrative purposes, we present one data analysis results in Section 6. Finally we conclude the paper in Section 6.

2 Maximum Likelihood Estimator

After re-ordering the data as necessary, we can assume without loss of generality, that the first \( n_1 \) observations are actual values, and the rest censored, so that we have the following observed data:

\[
\{(T_1, 1), \ldots, (T_{n_1}, 1), (L_{n_1+1}, R_{n_1+1}), \ldots, (L_{n_1+n_2}, R_{n_1+n_2})\},
\]

where \( n_1 + n_2 = n \). Thus, \( T_i \notin (L_i, R_i) \) for the first \( n_1 \) observations, while \( T_i \in (L_i, R_i) \) for the last \( n_2 \) observations. Based on the above information the likelihood function of the observed data can be obtained as follows. Note that, the likelihood function of the \( n \) observation is

\[
l(\theta) = c\theta^{n_1}e^{-\theta}\sum_{i=1}^{n_1} t_i \prod_{i=n_1+1}^{n_1+n_2} \left(e^{-\theta l_i} - e^{-\theta r_i}\right).
\]

(4)

Note that here \( c \) is the normalizing constant which depends on \( \alpha \) and \( \beta \), but since we are not interested in estimating \( \alpha \) and \( \beta \) at this moment, we are not making it explicit. Based on (4), the log-likelihood becomes

\[
\ln \theta = L(\theta) = \ln c + n_1 \ln \theta - \theta \sum_{i=1}^{n_1} t_i + \sum_{i=n_1+1}^{n_1+n_2} \ln \left(e^{-\theta l_i} - e^{-\theta r_i}\right).
\]

(5)

Taking the derivative of \( L(\theta) \) and putting equal to zero, we obtain

\[
\frac{\partial L}{\partial \theta} = \frac{n_1}{\theta} - \sum_{i=1}^{n_1} t_i + \sum_{i=n_1+1}^{n_1+n_2} \frac{(r_i - l_i)}{e^{\theta(r_i-l_i)} - 1} - \sum_{i=n_1+1}^{n_1+n_2} l_i = 0.
\]

(6)
Therefore, $\hat{\theta}$, the MLE of $\theta$, can be obtained by solving the equation (6). But it can not be obtained explicitly. We provide an iterative procedure to solve (6). Note that (6) can be written as

$$h(\theta) = \theta,$$  
(7)

where

$$h(\theta) = \frac{1}{\sum_{i=n_1+1}^{n_1+n_2} l_i + \sum_{i=1}^{n_1} t_i} \left[ n_1 + \theta \sum_{i=n_1+1}^{n_1+n_2} \frac{z_i e^{-\theta z_i}}{1 - e^{-\theta z_i}} \right].$$  
(8)

Therefore, a simple iterative procedure can be used to solve (7). For example, we can start with an initial guess $\theta^{(1)}$, then obtain $\theta^{(2)} = h(\theta^{(1)})$ and so on. The iterative procedure may be stopped if $|\theta^{(i)} - \theta^{(i+1)}| < \epsilon$, where $\epsilon$ is some preassigned small positive number. For an initial choice of $\theta$, we can use $\theta^{(1)} = n_1 / \sum_{i=1}^{n_1} t_i$.

Alternatively, EM algorithm also can be used to find the MLE of $\theta$. First let us obtain $E(T|L < T < R)$, where $L$ and $R$ fixed quantities and $T$ follows exponential with mean $\frac{1}{\theta}$.

Now

$$E(T|L < T < R) = \frac{e^{-\theta L} \left( L + \frac{1}{\theta} \right) - e^{-\theta R} \left( R + \frac{1}{\theta} \right)}{e^{-\theta L} - e^{-\theta R}}.$$  
(9)

Note that (9) can be used to compute the EM algorithm. The pseudo likelihood function will take the following form:

$$l(\theta) = \theta^{n_1+n_2} e^{-\theta \left( \sum_{i=1}^{n_1} T_i + \sum_{i=n_1+1}^{n_1+n_2} T_i^{(s)} \right)},$$  
(10)

where

$$T_i^{(s)} = \frac{e^{-\theta L} \left( L + \frac{1}{\theta} \right) - e^{-\theta R} \left( R + \frac{1}{\theta} \right)}{e^{-\theta L} - e^{-\theta R}}.$$  
(11)

Therefore, we use (9) for the ‘E’ step and then the ‘M’ step becomes quite trivial. The details are given below.

**EM Algorithm:**

- Step 1: Suppose $\theta^{(j)}$ is the $j^{th}$ iterate of $\hat{\theta}$.  

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• Step 2: Compute $T_{i(j)}^{(s)}$ by using (11) replacing $\theta$ by $\theta_{(j)}$.

• Step 3: $\theta_{(j+1)} = \frac{n_1 + n_2}{\sum_{i=1}^{n_1} T_i + \sum_{i=n_1+1}^{n_1+n_2} T_{i(j)}}$

3 THEORETICAL RESULTS

THEOREM 1: The iterative process provided in (7) will converge if

$$\sum_{i=n_1+1}^{n_1+n_2} r_i \leq 2 \sum_{i=1}^{n_1} t_i + 3 \sum_{i=n_1+1}^{n_1+n_2} l_i. \tag{12}$$

PROOF OF THEOREM 1: Consider

$$|h'(\theta)| = \frac{1}{\sum_{i=1}^{n_1} t_i + \sum_{i=n_1+1}^{n_1+n_2} l_i} \left| \sum_{i=n_1+1}^{n_1+n_2} z_i e^{-\theta z_i} \left(1 - e^{-\theta z_i} - \theta z_i\right) \right|. \tag{13}$$

Note that

$$\frac{|e^{-x}|(1 - e^{-x} - x)}{|1 - e^{-x}|^2} \leq \frac{1}{2} \quad \text{for all} \ x \geq 0,$$

therefore,

$$|h'(\theta)| \leq \frac{1}{2} \frac{\sum_{i=n_1+1}^{n_1+n_2} z_i}{\sum_{i=1}^{n_1} t_i + \sum_{i=n_1+1}^{n_1+n_2} l_i} \tag{14}$$

We know that the iterative process converges if $|h'(\theta)| < 1$, therefore, the result follows.

Now we need the following lemma to prove the consistency of the MLE.

LEMMA 1:

$$\frac{1}{n} L(\theta) \rightarrow g(\theta) \quad \text{a.s.} \tag{15}$$

where

$$g(\theta) = c + p(\theta_0) \ln \theta - \theta \left\{ \frac{1}{\theta_0} - \frac{(1 - p(\theta_0))(\alpha + \beta + 2\theta_0)}{(\alpha + \theta_0)(\beta + \theta_0)} \right\} - \theta \left(1 - p(\theta_0)\right) \cdot \left[ \frac{\alpha \beta}{\alpha + \theta_0} \sum_{i=1}^{\infty} \frac{1}{i(\beta + i\theta)} - \sum_{i=1}^{\infty} \frac{1}{i(\beta + i\theta + \theta_0)} \right].$$
and

\[ p(\theta) = \frac{\alpha \beta + \beta \theta + \theta^2}{(\alpha + \theta)(\beta + \theta)}. \]  

(13)

**Proof of Lemma 1:** Note that

\[
\frac{1}{n} L(\theta) = \frac{n_1}{n} \ln \theta - \frac{\theta}{n} \sum_{i=1}^{n_1} T_i - \frac{\theta}{n_{i+1}} \sum_{i=n_1+1}^{n_1+n_2} L_i + \frac{1}{n} \sum_{i=n_1+1}^{n_1+n_2} \ln \left(1 - e^{-\theta Z_i}\right).
\]

The density function of \( T \), conditional on the event that \( T \notin (L, R) \) can be written as

\[
f_{T \mid T \notin (L, R)}(t) = \frac{1}{p(\theta)} \left\{ \theta_0 e^{-(\alpha + \theta_0)t} + \theta_0 e^{-\theta_0 t} \left(1 - e^{-\alpha t} - \frac{\alpha e^{-\beta t}}{\alpha - \beta} \left(1 - e^{-(\alpha - \beta)t}\right)\right) \right\}
\]

if \( \alpha \neq \beta \)

and

\[
f_{T \mid T \notin (L, R)}(t) = \frac{1}{p(\theta)} \left\{ \theta_0 e^{-(\alpha + \theta_0)t} + \theta_0 e^{-\theta_0 t} \left(1 - e^{-\alpha t} - te^{-\alpha t}\right) \right\}
\]

if \( \alpha = \beta \).

Note that

\[ p(\theta) = P_\theta (T \notin (L, R)), \]

as defined in (13). Now

\[
E(T\mid T \notin (L, R)) = \frac{1}{p(\theta)} \left[ 1 - \frac{\theta_0}{(\alpha + \theta_0)^2} \right]
\]

if \( \alpha = \beta \),

\[
= \frac{1}{p(\theta)} \left[ \frac{\theta_0}{(\alpha + \theta_0)^2} + \frac{\theta_0}{\theta_0 - \frac{\alpha \theta_0}{(\alpha + \theta_0)^2}} - \frac{\alpha \theta_0}{\alpha - \beta} \left(\frac{1}{(\beta + \theta_0)^2} - \frac{1}{(\alpha + \theta_0)^2}\right)\right]
\]

if \( \alpha \neq \beta \).

Using the fact that the density function of \( L \) conditional on the event \( T \in (L, R) \) is

\[
f_{L \mid T \in (L, R)}(x) = \frac{1}{1 - p(\theta)} \times \frac{\alpha \theta_0}{(\beta + \theta_0)} e^{-(\alpha + \theta_0)x}, \quad \text{for } x > 0,
\]

we have,

\[
E(L\mid T \in (L, R)) = \frac{1}{1 - p(\theta)} \times \frac{\alpha \theta_0}{(\beta + \theta_0)(\alpha + \theta_0)^2}.
\]

Similarly, since the density function of \( Z \) conditioned on \( T \in (L, R) \) is

\[
f_{Z \mid T \in (L, R)}(z) = \frac{1}{1 - p(\theta)} \times \frac{\alpha \beta e^{-\beta z}}{(\alpha + \theta_0)} (1 - e^{-\theta_0 z}), \quad \text{for } z > 0,
\]

we have,
therefore,

\[
E \left( \ln \left( 1 - e^{-\theta_0 z} \right) \right) = -\frac{1}{1 - p(\theta_0)} \times \frac{\alpha \beta}{(\alpha + \theta_0)} \left[ \sum_{i=1}^{\infty} \frac{1}{i(\beta + i\theta)} - \sum_{i=1}^{\infty} \frac{1}{i(\beta + i\theta + \theta_0)} \right].
\]

Now the result follows using \( \frac{\sum}{n} \to p(\theta_0) \) and the strong law of large numbers.

**LEMMA 2:** \( g(\theta) \) is a unimodal function, with a unique maximum.

**PROOF OF LEMMA 2:** It follows from the fact \( g'(0) = \infty \), \( g'(\infty) < 0 \) and \( g''(0) < 0 \).

**LEMMA 3:** The MLE of \( \theta_0 \), say \( \hat{\theta} \), will converge to \( \theta^* \), where \( \theta^* \) is a solution of the non-linear equation

\[
h(\theta) = \frac{p(\theta_0)}{\theta} - \frac{1}{\theta_0} + \frac{(1 - p(\theta_0))(\alpha + \beta + 2\theta_0)}{(\alpha + \theta_0)(\beta + \theta_0)} - \frac{1 - p(\theta_0)}{(\alpha + \theta_0)}
\]

\[
- \frac{\alpha \beta}{(\alpha + \theta_0)} \left[ \sum_{i=1}^{\infty} \frac{1}{(\beta + \theta_0 + i\theta)^2} - \sum_{i=1}^{\infty} \frac{1}{(\beta + i\theta)^2} \right] = 0, \quad (14)
\]

where \( p(\theta) \) is as defined before.

**PROOF OF LEMMA 3:** In this particular proof we denote \( \hat{\theta} \) by \( \hat{\theta}_n \)

Case 1: \( \hat{\theta}_n \) is bounded for all \( n \).

Suppose \( \hat{\theta}_n \) does not converge to \( \theta^* \). Therefore, there exists a subsequence \( \{n_k\} \) of \( \{n\} \) and \( \tilde{\theta} \neq \theta^* \), such that \( \hat{\theta}_{n_k} \to \tilde{\theta} \). Since \( \hat{\theta}_{n_k} \) is the MLE,

\[
\frac{1}{n_k} L(\hat{\theta}_{n_k}) \geq \frac{1}{n_k} L(\theta^*)
\]

Taking limits on both sides of (3) we get

\[
g(\tilde{\theta}) \geq g(\theta^*),
\]

which leads to a contradiction because \( \theta^* \) is the unique maximum of \( g(\theta) \).

Case 2: \( \hat{\theta}_n \) is not bounded.
In this case there exists a subsequence \(\{n_k\}\) of \(\{n\}\) such that \(\hat{\theta}_{n_k} \to \infty\). Note that

\[
\frac{1}{n_k}L(\hat{\theta}_{n_k}) \geq \frac{1}{n_k}L(\theta^*),
\]

and as \(\hat{\theta}_{n_k} \to \infty\), \(\frac{1}{n_k}L(\hat{\theta}_{n_k}) \to -\infty\). Since \(\frac{1}{n_k}L(\theta^*)\) converges to a fixed number, it leads to a contradiction.

Now since \(\theta_0\) is a solution of (14), we have

**Theorem 2:** The MLE of \(\theta\) is a consistent estimator of \(\theta_0\).

Now we provide the asymptotic distribution of the MLE.

**Theorem 3:** The maximum likelihood estimator has the following asymptotic distribution

\[
\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \frac{\sigma^2}{c^2}),
\]

where

\[
\sigma^2 = \left[ E \left\{ \left( T - \frac{1}{\theta_0} \right)^2 \bigg| T \notin (L, R) \right\} - \left\{ E \left( T - \frac{1}{\theta_0} \bigg| T \notin (L, R) \right) \right\} \right]^2 + \left[ E(L^2\big|T \in (L, R)) - (E(L\big|T \in (L, R)) \right]^2
\]

\[
+ \left[ E \left\{ \left( \frac{Ze^{-\theta_0 Z}}{1 - e^{-\theta_0 Z}} \right)^2 \bigg| T \in (L, R) \right\} - \left\{ \left( \frac{Ze^{-\theta_0 Z}}{1 - e^{-\theta_0 Z}} \bigg| T \in (L, R) \right) \right\} \right]^2
\]

and

\[
c = \frac{p(\theta_0)}{\theta_0^2} + (1 - p(\theta_0)) \left\{ E \left( \frac{Z^2 e^{-\theta_0 Z}}{(1 - e^{-\theta_0 Z})^2} \bigg| T \in (L, R) \right) \right\}.
\]

To prove Theorem 3, we need the following lemma;

**Lemma 4:** Suppose \(U_i\)'s are sequence of independent and identically distributed random variables with \(E(U_1) = 0\), \(V(U_1) = 1\) and \(\{N(n)\}\) is a sequence of discrete random variables with the following probability density function:

\[
P(N(n) = i) = \binom{n}{i} p^i (1 - p)^{n-i}; \quad i = 0, \ldots, n,
\]
where $0 < p < 1$. Then as $n \to \infty$,

$$\frac{1}{\sqrt{N(n)}} \sum_{i=1}^{N(n)} U_i \overset{d}{\to} N(0, 1).$$

**Proof of Lemma 4:** Suppose

$$Y_{N(n)} = \frac{1}{\sqrt{N(n)}} \sum_{i=1}^{N(n)} U_i$$

and the characteristic function of $Y_{N(n)}$ is $\phi_{N(n)}(t)$. Then,

$$\phi_{N(n)}(t) = \mathbb{E}\left(e^{itY_{N(n)}}\right) = \mathbb{E}\left(\left(\frac{e^{it\frac{1}{\sqrt{N(n)}} \sum_{i=1}^{N(n)} U_i}}{e^{it\frac{1}{\sqrt{N(n)}} \sum_{i=1}^{N(n)} U_i}}\right)\right) \sum_{k=0}^{n} \left(\frac{1}{\sqrt{k}}\right)^k \mathbb{P}(N(n) = k) \left(\frac{n}{k}\right) p^k(1 - p)^{n-k}.$$

Now if $\phi_U(.)$ denotes the characteristic function of $U_1$, then for fixed $t$,

$$\left|\phi_{N(n)}(t) - e^{-t^2/2}\right| \leq \sum_{k=0}^{n} \left|E\left(\left(e^{it\frac{1}{\sqrt{k}} \sum_{i=1}^{k} U_i}\right)\mathbb{P}(N(n) = k) - e^{-t^2/2}\right) \left(\frac{n}{k}\right) p^k(1 - p)^{n-k}\right|$$

$$= \sum_{k=0}^{n} \left|\phi_U\left(\frac{t}{\sqrt{k}}\right)^k - e^{-t^2/2}\right| \left(\frac{n}{k}\right) p^k(1 - p)^{n-k}.$$

Since by Central Limit Theorem

$$\lim_{k \to \infty} \phi_U\left(\frac{t}{\sqrt{k}}\right)^k = e^{-t^2/2},$$

therefore, for a given $\epsilon > 0$, choose $N_1(t)$ large enough so that for $k \geq N_1(t)$

$$\left|\phi_U\left(\frac{t}{\sqrt{k}}\right)^k - e^{-t^2/2}\right| \leq \epsilon.$$

Moreover, for fixed $N_1(t)$, choose $n$ large enough so that

$$\sum_{i=0}^{N_1(t)} \left(\frac{n}{i}\right) p^i(1 - p)^{n-i} \leq \epsilon.$$
Therefore,
\[
\left| \hat{\phi}_{N(n)}(t) - e^{\frac{-t^2}{2}} \right| \leq \sum_{k=0}^{N_1(t)} \left| \left( \phi_U \left( \frac{t}{\sqrt{k}} \right) \right)^k - e^{\frac{-t^2}{2}} \left( \binom{n}{k} p^k (1-p)^{n-k} \right) \right| + \sum_{k=N_1(t)+1}^{n} \left| \left( \phi_U \left( \frac{t}{\sqrt{k}} \right) \right)^k - e^{\frac{-t^2}{2}} \left( \binom{n}{k} p^k (1-p)^{n-k} \right) \right|
\leq 2\epsilon + \epsilon = 3\epsilon.
\]

Since \( \epsilon \) is arbitrary, the result follows from the fact that \( e^{\frac{-t^2}{2}} \) is the characteristic function of \( N(0,1) \) random variable.

**Proof of Theorem 3.** Note that
\[
L(\theta) = n_1 \ln \theta - \theta \sum_{i=1}^{n_1} T_i - \theta \sum_{i=n_1+1}^{n_1+n_2} L_i + \sum_{i=n_1+1}^{n_1+n_2} \ln \left( 1 - e^{-\theta Z_i} \right),
\]
and
\[
L'(\theta) = \frac{n_1}{\theta} - \sum_{i=1}^{n_1} T_i - \sum_{i=n_1+1}^{n_1+n_2} L_i + \sum_{i=n_1+1}^{n_1+n_2} \frac{Z_i e^{-\theta Z_i}}{(1 - e^{-\theta Z_i})},
\]
Using mean value theorem,
\[
L'(\hat{\theta}) - L'(\theta_0) = (\hat{\theta} - \theta_0) L''(\bar{\theta}),
\]
where \( \bar{\theta} \) is a point between \( \hat{\theta} \) and \( \theta_0 \). Therefore,
\[
\sqrt{n}(\hat{\theta} - \theta_0) = -\frac{1}{\sqrt{n}} L'(\theta_0) \cdot \frac{1}{\left( \sqrt{n} L''(\bar{\theta}) \right)}.
\]
Now the proof will be complete once we show that;
\[
\frac{1}{\sqrt{n}} L'(\theta_0) \xrightarrow{d} N(0,\sigma^2) \tag{15}
\]
and
\[
\frac{1}{n} L''(\theta) \xrightarrow{a.s.} c, \tag{16}
\]
here ‘a.s.’ means convergence almost surely. Now note that (15) follows from Lemma 4. The proof of (16) follows from the fact that \( \bar{\theta} \) converges to \( \theta_0 \) a.s and from the strong law of large numbers.
In this section we consider a Bayesian formulation of the problem of estimating the parameter \( \theta \). We will assume that the parameter \( \theta \) has a gamma prior distribution with the shape parameter \( a \) and scale parameter \( b \) – denoted by \( \text{Gamma}(a, b) \). The density function of the prior density of \( \theta \) for \( a, b > 0 \), is

\[
\pi(\theta) = \pi(\theta|a, b) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}.
\] (17)

No prior distribution on the censoring parameters are assumed. Based on the above assumption, the likelihood function of the observed data is

\[
l(data|\theta) = c \theta^{n_1} e^{-\theta \sum_{i=1}^{n_1} t_i} \prod_{i=n_1+1}^{n_1+n_2} (1 - e^{-\theta z_i}) e^{-\theta \sum_{i=n_1+1}^{n_1+n_2} l_i}.
\] (18)

By a slight abuse of the notation, writing \( z_i = z_{n_1+i} \) and \( l_i = l_{n_1+i} \) we can rewrite (18) as

\[
l(data|\theta) = c \theta^{n_1} e^{-\theta \sum_{i=1}^{n_1} t_i} \prod_{i=1}^{n_2} (1 - e^{-\theta z_i}) e^{-\theta \sum_{i=1}^{n_2} l_i}.
\] (19)

Based on (17), the joint density of the data and \( \theta \) is

\[
l(data|\theta)\pi(\theta).
\] (20)

Based on (20), we obtain the posterior density of \( \theta \) given the data as

\[
\pi(\theta|data) = \frac{l(data|\theta)\pi(\theta)}{\int_0^\infty l(data|\theta)\pi(\theta)d\theta}.
\] (21)

Now we compute the numerator of the right hand side of (2). We can write

\[
l(data|\theta)\pi(\theta) = c \theta^{a+n_1-1} e^{-\theta(b + \sum_{i=1}^{n_1} t_i + \sum_{i=1}^{n_2} l_i)} \prod_{i=1}^{n_2} (1 - e^{-\theta z_i}).
\] (22)

Note that

\[
(1 - e^{-\theta z_i}) = \sum_{P_j} (-1)^{|P_j|} e^{-\theta(z,P_j)}.
\] (23)
where $P_j$ is a vector length $n_2$ and each entries of $P_j$ are either 0 or 1. $|P_j|$ denotes the sum of elements of $P_j$ and $z = (z_1, \ldots, z_{n_2})$. The summation on the right hand side of (23) is over $2^{n_2}$ elements and $(z.P_j)$ denotes the usual dot product between the two vectors of equal lengths. Using (23), (2) can be written as

$$l(data|\theta)\pi(\theta) = c \sum_{P_j} (-1)^{|P_j|} \theta^{a+n_1-1} e^{-\theta(b+\sum_{i=1}^{n_1} t_i+\sum_{i=1}^{n_2} l_i+(z.P_j))}. \quad (24)$$

So we obtain

$$\int_0^\infty l(data|\theta)\pi(\theta)d\theta = c \sum_{P_j} (-1)^{|P_j|} \frac{\Gamma(a+n_1)}{(b+\sum_{i=1}^{n_1} t_i+\sum_{i=1}^{n_2} l_i+(z.P_j))^{a+n_1-1}}. \quad (25)$$

Therefore, the posterior density of $\theta$ given the data for $\theta > 0$, is

$$\pi(\theta|data) \propto \frac{\sum_{P_j} (-1)^{|P_j|} \theta^{a+n_1-1} e^{-\theta(b+\sum_{i=1}^{n_1} t_i+\sum_{i=1}^{n_2} l_i+(z.P_j))}}{\sum_{P_j} (b+\sum_{i=1}^{n_1} t_i+\sum_{i=1}^{n_2} l_i+(z.P_j))^{a+n_1-1}}. \quad (26)$$

Therefore, the Bayes estimate of $\theta$ under squared error loss function is

$$E(\theta|data) = \frac{\sum_{P_j} (-1)^{|P_j|} \theta^{a+n_1-1} e^{-\theta(b+\sum_{i=1}^{n_1} t_i+\sum_{i=1}^{n_2} l_i+(z.P_j))}}{\sum_{P_j} (b+\sum_{i=1}^{n_1} t_i+\sum_{i=1}^{n_2} l_i+(z.P_j))^{a+n_1-1}}. \quad (27)$$

When $n_2$ is small the evaluation of $E(\theta|data)$ is not difficult, but for large $n_2$ it is difficult to compute numerically. We propose a simple Gibbs sampling technique to compute $E(\theta|data)$ and for constructing the corresponding credible interval. Note that when $n_2 = 0$, then,

$$\pi(\theta|data) \sim Gamma(a + n_1, b + \sum_{i=1}^{n_1} t_i); \quad (28)$$

as should be expected. Moreover, the conditional density of $T$, given $T \in (L, R)$, is

$$f_{T|T \in (L,R)}(x|\theta) = \frac{\theta e^{-\theta x}}{e^{-\theta L} - e^{-\theta R}} \quad \text{if} \quad L < x < R. \quad (29)$$

Using (28) and (29) we propose the following Gibbs sampling scheme to generate $\theta$ from its posterior distribution.
**Gibbs Sampling Scheme:**

Step 1: Generate $\mu_{1,1}$ from $\text{Gamma}(a + n_1, b + \sum_{i=1}^{n_1} t_i)$.

Step 2: Generate $t^{(n_1+i)}$ for $i = 1, \ldots, n_2$ from $f_{T|R}(u_{n_1+i}, r_{n_1+i} | \cdot | \theta_{1,1})$.

Step 3: Generate $\theta_{2,1}$ from $\text{Gamma}(a + n_1 + n_2, b + \sum_{i=1}^{n_1} t_i + \sum_{i=n_1+1}^{n_1+n_2} t^{(i)})$.

Step 4: Go back to Step 2, and replace $\theta_{1,1}$ by $\theta_{2,1}$ and repeat Steps 2 and 3 for $N$ times.

From the generated $N$, $\theta_{2,j}$, the Bayes estimate of $\theta_0$, under squared error loss function can be be computed as

$$\frac{1}{N-M} \sum_{j=M+1}^{N} \theta_{2,j}$$

(30)

where $M$ is the burn-in sample. Similarly, using the method of Chen and Shao (1999), the highest posterior density (HPD) credible interval of $\theta_0$ also can be constructed.

5 Numerical Results

In this section we mainly compare how the different methods work for small sample sizes and for different censoring scheme. All the simulations are performed at the Department of Mathematics and Statistics, IIT Kanpur using the Pentium-IV machine. We have used the random number generator RAN2 of Press et al. (1992) for our simulation purposes. The programming language FORTRAN-77 has been used and it can be obtained on request from the authors. All the results are based on 1000 replications.

We considered different sample sizes namely $n = 10, 20, 30, 40, 50$ and different censoring schemes. For the censoring scheme we considered the following combinations of $(1/\alpha, 1/\beta) = (0.5, 0.25), (0.5, 0.5), (0.5, 0.75), (1.25, 0.25), (1.25, 0.50)$ and $(1.25, 0.75)$. From the given sample we compute maximum likelihood estimator of $\theta$ using the EM algorithm and also using the iterative method proposed in section 2. It is observed in both cases they converge to the same value. We also compute the 95% confidence intervals based on the
asymptotic distribution of the maximum likelihood estimator and replacing the expected Fisher information by the empirical Fisher information. Meeker and Escobar (1998) reported that the confidence interval based on the asymptotic distribution of \( \ln \hat{\theta} \) is usually superior to one of \( \hat{\theta} \). We computed the confidence interval based on the asymptotic distribution of \( \ln \hat{\theta} \). For comparison purposes, the Bayes estimates under squared error loss function and the corresponding 95% Monte Carlo HPD credible interval as suggested by Chen and Shao (1999) are also reported in Tables 1 and 2. All the Bayes estimates are computed using the prior \( a = 0 \) and \( b = 0 \). Note that the above prior is non-informative and non-proper prior. Although, the prior is non-proper but the corresponding posterior has a proper density function. As suggested by Congdon (2001), we tried the prior \( a = 0.0001 \) and \( b = 0.0001 \), which is a proper prior but which is almost non-informative, the results are not significantly different and they are not reported here.

From Table 1 it is clear that as the sample size increases for all the censoring scheme, the average biases and mean squared errors decrease for both the maximum likelihood estimator and Bayes estimator. It verifies the consistency properties of both the estimators. For fixed sample size and for fixed \( \alpha \), as \( 1/\beta \) increases (severe censoring), the biases and the mean squared errors both increase for the maximum likelihood estimates, interestingly, for the Bayes estimates although the mean squared errors decrease but the same can not be said for the biases. Otherwise, they behave quite similarly both in terms of biases and mean squared errors.

From Table 2 it is clear that for all the three suggested methods as the sample size increases, the average lengths of the confidence/credible intervals decrease. Similarly, for fixed sample size and for fixed \( \alpha \) as \( 1/\beta \) increases, the average lengths increase as expected. For all the three cases, the coverage percentages are quite close to the nominal level (95%) even when the sample size is as small as 10. The performances of all the methods are quite
similar in nature. For very small sample sizes (namely 10), the Bayes credible intervals have marginally smaller magnitudes than the asymptotic confidence intervals, but for moderate sample sizes (namely 20, 30 and 40) it is the other way. The average confidence intervals based on the transformed maximum likelihood estimators (MEE) have larger lengths compared to the other two.

6 Data Analysis

For illustrative purposes, we present a real data analysis results using our proposed method. The data set is taken from Lawless (1982, p. 491). It consists of failure times for 36 appliances subject to an automatic life tests. Although the original data has also the cause of failure with each failure time, but here we are interested about the overall failure distribution and we do not consider the cause of failure in this case. This data set was analyzed using exponential and Weibull models by Kundu and Basu (2000) and it was observed that the exponential model can be used instead of Weibull model. For the complete data set it is observed that the maximum likelihood estimate of $\theta_0$ is 0.00036. The Kolmogorov-Smirnov distance between the empirical distribution function and the fitted exponential distribution function is 0.1944 and the corresponding $p$ value is 0.1317. Therefore, exponential model can not be rejected.

Now we created an artificial data by middle censoring, whose left end was an exponential random variable with mean 500 and the width was exponential with mean 1000. The data after rearranging are presented below:

Data set: 11, 35, 49, 170, 958, 1062, 1167, 1594, 1925, 1990, 2223, 2327, 2400, 2451, 2471, 2551, 2565, 2568, 2694, 2761, 2831, 3034, 3059, 3112, 3214, 3478, 3504, 4329, 6367, 6976, 7846, 13403, (118.66, 1224.04), (377.76, 2011.51), (351.65, 720.48), (125.96, 4226.08).

The summary statistics of the data are as follows: $n = 36$, $n_1 = 32$, $n_2 = 4$, $\sum_{i=1}^{n_1} t_i = 95125$, $\sum_{i=n_1+1}^{n_2} r_i = 8182.11$, $\sum_{i=n_1+1}^{n_2} l_i = 974.03$. Therefore, the iterative process starts with
the initial guess \( \theta^{(1)} = 32/95125 = 0.000336 \). Since \( r_i, t_i \) and \( l_i \) satisfy the condition (12) of Theorem 1, therefore, the proposed iterative process will converge. The log-likelihood surface without the additive constant is provided in Figure 1. It clearly shows that the log-likelihood surface is an unimodal function. Therefore, the EM algorithm also should not have any problem of convergence. The iterative process (7) stops after three iteration and the solution is 0.000364. The 95% confidence intervals based on the asymptotic distribution of \( \hat{\theta} \) and \( \ln \hat{\theta} \) are (0.00024, 0.00048) and (0.00026, 0.00051) respectively. The Bayes estimate (the posterior mean) under the non-informative and non-proper prior becomes 0.000362 and the corresponding 95% HPD credible interval is (0.00025, 0.00049). The histogram of the generated posterior sample and the fitted gamma distribution are presented in Figure 2. In the same figure we have also plotted the fitted posterior density function assuming \( n_2 = 0 \). It shows the posterior information of the censored observations.
Figure 2: Histogram of the generated 1000 posterior sample and the fitted posterior density functions.

7 CONCLUSIONS

In this paper we have considered the parametric inference of the middle censored data when the lifetime distributions are exponential. We have developed inferential procedures for both the classical and Bayesian set up. Although this paper mainly address the problem when the lifetime distributions are exponential but similar inferential procedures can be developed for other lifetime distributions also, for example Weibull, gamma, log-normal distributions etc. More work is needed along that direction.

References


Table 1: The average estimates and the corresponding mean squared errors (within brackets) are reported for the different estimators.

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Table 2: The average lengths of the confidence/credible intervals and the corresponding coverage percentages (within brackets) are reported.

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