A class of nonpositively curved Kahler manifolds biholomorphic to the unit ball in $\mathbb{C}^n$

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A CLASS OF NONPOSITIVELY CURVED KÄHLER MANIFOLDS
BIHOLOMORPHIC TO THE UNIT BALL IN $\mathbb{C}^n$

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Abstract. Let $(M, g)$ be a simply connected complete Kähler manifold with nonpositive sectional curvature. Assume that $g$ has constant negative holomorphic sectional curvature outside a compact set. We prove that $M$ is then biholomorphic to the unit ball in $\mathbb{C}^n$, where $\dim_{\mathbb{C}} M = n$.

1. Introduction

An important issue in complex differential geometry is to understand the relationship between the curvature of a Kähler manifold and the underlying complex structure. The simplest theorem along these lines is the classical theorem of Cartan that a simply connected complete Kähler of constant holomorphic sectional curvature is holomorphically isometric to $\mathbb{C}P^n$, $\mathbb{B}^n$ or $\mathbb{C}^n$ (where $\mathbb{B}^n$ is the open unit ball in $\mathbb{C}^n$) depending on whether the curvature is positive, negative or zero. Here the metrics on $\mathbb{C}P^n$, $\mathbb{B}^n$ or $\mathbb{C}^n$ are the Fubini-Study metric, the Bergman metric and the flat metric, respectively.

A far deeper theorem is that of Siu and Yau [7] which states that a complete simply connected nonpositively curved Kähler manifold of faster than quadratic curvature decay has to be biholomorphic to $\mathbb{C}^n$. An analogue of this theorem for characterizing the ball in $\mathbb{C}^n$ is not known, to the best of our knowledge. As a first step in this direction we prove the following theorem, which can also be regarded as a perturbed version of Cartan’s theorem stated above, at least in the negative case.

Theorem 1.1. Let $(M, g)$ be a simply connected complete Kähler manifold with nonpositive sectional curvature. If $g$ has constant negative holomorphic sectional curvature outside a compact set, $M$ is biholomorphic to the unit ball in $\mathbb{C}^n$, where $\dim_{\mathbb{C}} M = n$.

It is natural to ask if the theorem is true if we only have that the holomorphic sectional curvatures converge to $-1$ as $r \to \infty$, where $r$ is the distance from a fixed point in $M$. However, the following class of examples show that the theorem then fails: If $g$ is the Bergman metric of a strongly pseudoconvex domain $\Omega$ in $\mathbb{C}^n$, then the holomorphic sectional curvatures of $g$ approach $-1$ as one approaches $\partial \Omega$. Moreover, by the results of [5], if $\Omega$ is a small enough perturbation of $\mathbb{B}^n$, i.e., $\partial \Omega$ is a $C^\infty$ small perturbation of $\partial \mathbb{B}^n$, then $g$ has negative sectional curvature. By
Chern-Moser theory, “many” of these perturbations are not biholomorphic to $\mathbb{B}^n$. Hence one should impose a specified rate of convergence of holomorphic sectional curvatures to $-1$ in order to obtain a theorem similar to that of Siu-Yau in the negative case. However, it is not clear what this rate of convergence should be.

Finally, we note that the hypotheses of the Siu-Yau theorem are strong enough to guarantee that the Kähler manifold is actually holomorphically isometric to $\mathbb{C}^n$ with the flat metric. In fact, R. Greene and H. Wu proved that a Riemannian manifold with the same curvature hypotheses has to be flat [6]. In our case, however, we can always perturb the Bergman metric of $\mathbb{B}^n$ on a compact set and satisfy our hypotheses.

Roughly, the proof of Theorem 1.1 proceeds as follows: Suppose that $M$ has constant holomorphic curvature outside a compact set $K$. As in the proof of Cartan’s theorem, one can use the exponential map to construct holomorphic maps to $\mathbb{B}^n$ on “pieces” of $M \setminus K$. The difficulty here is that even though these maps can be chosen to patch up to a give single holomorphic map from $M \setminus K$ to $\mathbb{B}^n$, this map may not be injective. We avoid this difficulty by working with $\partial M$, the asymptotic boundary of $M$. More precisely, we use the holomorphic maps above to define a spherical CR-structure on $\partial M$. Since $\partial M$ is simply connected, one gets a global CR-diffeomorphism to $S^{2n-1}$. One then notes that since $M$ is Stein, we can extend this diffeomorphism to $M$ by Hartogs’ theorem.

2. Proof

For the rest of this paper, $M$ will denote a simply-connected, complete Kähler manifold with nonpositive sectional curvature and constant holomorphic sectional curvature $-1$ outside a compact set. $\partial M$ will denote its asymptotic boundary. There is a natural topology, described in the proof below, on $\overline{M} := M \cup \partial M$ which makes it a compact topological manifold-with-boundary.

The main theorem is proved by first proving the following proposition. In what follows $S^{2n-1}$ is the unit sphere in $\mathbb{C}^n$ with the induced CR-structure.

Lemma 2.1. $\overline{M}$ can be given the structure of a smooth compact manifold-with-boundary such that $\partial M$ admits a “natural” CR-structure which makes it CR-diffeomorphic to $S^{2n-1}$.

Before beginning the proof, we recall certain general constructions on nonpositively curved manifolds:

First, we define the “modified” exponential map. Let $V$ be a complex vector space with a Hermitian inner product $h$ and let $B = \{x \in V : \|x\| < 1\}$ denote the open unit ball in $V$. Define the homeomorphism $\phi : V \to B$ by $\phi(x) = (1 - e^{-\|x\|}) \frac{x}{\|x\|}$. Note that $\phi$ is a diffeomorphism on $V \setminus \{0\}$. When $V = T_pM$ and $h = g_p$, we will use the notation $\phi_p$. For any $p \in M$, define the modified exponential map $\overline{\exp}_p : B_p \to M$ by $\overline{\exp}_p = \exp_p \circ \phi_p^{-1}$.

Next, let $\partial M = \{\text{equivalence classes of geodesic rays in } M\}$, where geodesics $c_1, c_2 : [0, \infty) \to M$ are equivalent if there is a constant $a < \infty$ such that $d(c_1(t), c_2(t)) < a$ for all $t \geq 0$. $\partial M$ is usually referred to as the asymptotic boundary of $M$. We endow $\overline{M} = M \cup \partial M$ with the “cone” topology. This is the topology generated by open sets in $M$ and “cones”, corresponding to $x \in M$. 

$z \in \partial M$ and $\varepsilon > 0$, defined by
\[ C_x(z, \varepsilon) := \{ y \in M \mid y \neq x \text{ and } <x (z, y) < \varepsilon \}. \]

Here the angle $<x (z, y) := < (c_1(0), c_2(0))$ where $c_1$ and $c_2$ are geodesics joining $x$ with $z$ and $y$ (see [1] for details). For any $p \in M$, $\exp_p$ extends to a homeomorphism, which we continue to denote by the same symbol, from $\overline{B}_p$ to $\overline{M}$.

Now we come to the proof of Lemma 2.1.

**Proof.** Suppose $M$ has constant holomorphic curvature $-1$ outside a compact set $K$. Fix $o \in M$. Choose $R$ large so that $d(o, x) < R$ for any $x \in K$.

If $p \in \partial M$, then there is an unit-speed geodesic $\gamma_p : [0, \infty) \to M$ with $\gamma_p(0) = o$ and $\lim_{t \to -\infty} \gamma_p(t) = p$. Let $x(p) := \gamma_p(R)$. We observe that $C_x(p, \frac{\pi}{4}) \cap K = \emptyset$. This is because $d(o, x) > R$ for any $x \in C_x(p, \frac{\pi}{4})$ by Toponogov’s Comparison Theorem for geodesic triangles in nonpositively curved manifolds (see [1], Page 5). Hence, by our choice of $R$, $g$ has constant holomorphic sectional curvature in the interior of $C_x(p, \frac{\pi}{4})$.

Choose $p_1, .., p_k \in \partial M$ so that if $U_i := C_{x(p_i)}(p_i, \frac{\pi}{4})$, then $U_1 \cap \partial M, .., U_k \cap \partial M$ cover $\partial M$.

For $i = 1, .., k$, choose linear isometries $L_i : T_{\gamma_p(R)} \to T_0 \mathbb{B}^n$. We then get maps
\[ f_i := \exp_p \circ L_i \circ \exp_p^{-1} : U_i \to \overline{\mathbb{B}}^n. \]

These maps are homeomorphisms onto their images and we declare these to be charts on $\partial M \subset M$. In order to check that the transition functions are smooth, let us observe the following:

First, it is easily checked that $f_i|U_i = \exp_0 \circ L_i \circ \exp_0^{-1}$. Recall that our metric is locally symmetric in the interior of $U_i$. Hence by the Cartan-Ambrose-Hicks Theorem (cf. [4]), $f_i$ is a holomorphic local isometry there.

Next, Toponogov’s Comparison Theorem implies that for any $p \in \partial M$, $U_p$ is geodesically convex. Also, it is clear from the definition that if $q \in U_p$, then the geodesic ray starting at $x(p)$ and passing through $q$ lies in $U_q$. Combining these observations, we see that if $U_i \cap U_j \neq \emptyset$, then

\[ U_i \cap U_j \text{ is connected and } U_i \cap U_j \cap \partial M \neq \emptyset. \]  

(2.1)

Now the transition function $f_i \circ f_j^{-1}$ is a holomorphic isometry (for the restriction of the Bergman metric of $\mathbb{B}^n$) from $f_j(U_i \cap U_j) \subset \mathbb{B}^n$ to $f_i(U_i \cap U_j) \subset \mathbb{B}^n$. Since $f_j(U_i \cap U_j)$ is connected by (2.1), such a mapping has to be the restriction of a global automorphism of $\mathbb{B}^n$. In particular, the mapping is smooth up to the boundary, i.e. $f_i \circ f_j^{-1} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ is smooth. This gives us the required smooth structure on $\overline{M}$.

Also, it is clear that the charts $(U_i \cap \partial M, f_i)$ define a CR-structure on $\partial M$, since the transition functions will be local CR-diffeomorphisms of $S^{2n-1}$. Moreover, by definition this CR-structure is locally spherical. Since $\partial M$ is compact and simply connected, the results of [2] (basically a developing map argument) imply that there is a global diffeomorphism $\psi$ from $\partial M$ to $S^{2n-1}$. This proves the lemma. □
We continue with the proof of the main theorem, using the notation in the proof of the lemma. Let us note that by composing with holomorphic automorphisms of $\mathbb{B}^n$, if necessary, we can assume that

$$f_i|_{U_i \cap \partial M} = \psi.$$  

By (2.1), (2.2) and unique continuation, if $U_i \cap U_j \neq \emptyset$ then $f_i \circ f_j^{-1} = \text{id} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$, since $f_i \circ f_j^{-1} = \text{id}$ on $f_j(U_i \cap U_j) \cap \partial \mathbb{B}^n$.

Hence if $U_i \cap U_j \neq \emptyset$, then $f_i = f_j$ on $U_i \cap U_j$. Therefore, the $f_i$ patch up to give a smooth mapping $F : U \to \mathbb{B}^n$ on the open neighbourhood $U = U_1 \cup \ldots \cup U_k$ of $\partial M$ and $F$ is holomorphic on $U \cap M$. Since $F|_{\partial M} = \psi$ is injective and since $F$ is a local diffeomorphism, we can choose a neighbourhood $V \subset U$ of $\partial M$ such that $F|_V$ is injective.

To extend $F$ to $M$, we recall Wu’s theorem [8] that a simply connected complete Kähler manifold of nonpositive sectional curvature is Stein. Combining this with the fact that $M \setminus U$ is compact, we conclude that $F$ extends to all of $M$ by Hartogs’ theorem on Stein manifolds. By the maximum principle, $F(M) \subset \mathbb{B}^n$.

To construct the inverse of $F$, let $G = F^{-1}|_V$. $G$ is smooth map defined on the neighbourhood $F(V)$ of $\partial \mathbb{B}^n$, which is holomorphic in $F(V) \cap \mathbb{B}^n$. Since $M$ is Stein, $G$ is an embedded submanifold of some $\mathbb{C}^N$. Again by Hartogs theorem and the maximum principle, $G : F(V) \cap \mathbb{B}^n \to V \subset \mathbb{C}^N$ extends to a smooth map $G : \mathbb{B}^n \to \bar{M}$ which is holomorphic in $\mathbb{B}^n$.

Finally, by unique continuation, $F \circ G = \text{id}_{\mathbb{B}^n}$ and $G \circ F = \text{id}_{\mathbb{B}^n}$. Q.E.D.

**Remark:** A CR-structure on the boundary of a nonpositively curved Kähler manifold is shown to exist under hypotheses more general than ours in [3].

We end with the following

**Question:** Let $(M^n, g)$ be a simply-connected complete Kähler manifold of nonpositive curvature. If $g$ is locally symmetric outside a compact set, is $M$ biholomorphic to $\Omega \times \mathbb{C}^{n-k}$, where $\Omega$ is a bounded symmetric domain in $\mathbb{C}^k$, for some $k$?

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**References**


List of Modifications:

1. Added title in French on page 1.

2. In the “Résumé”, “sectionelle” changed to “sectionnelle”.

3. Fourth line from bottom of first page, $\partial M$ changed to $\partial \Omega$.

4. First line on page 3, :=< changed to := <.

5. On page 3, “..by Toponogov..” changed to “..Toponogov..”.


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