On the compactness of isometry groups in complex analysis

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ON THE COMPACTNESS OF ISOMETRY GROUPS IN COMPLEX ANALYSIS

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Abstract. We prove that the group of continuous isometries for the Kobayashi or Carathéodory metrics of a strongly convex domain in $\mathbb{C}^n$ is compact unless the domain is biholomorphic to the ball.

A key ingredient, proved using differential geometric ideas, is that a continuous isometry between a strongly convex domain and the ball has to be biholomorphic or anti-biholomorphic. Combining this with a metric version of Pinchuk rescaling then gives the main result.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. Suppose that there exists a sequence $\phi_\nu : \Omega \to \Omega$, $\nu \geq 1$, of holomorphic automorphisms and a point $z_0 \in \Omega$ such that $\{\phi_\nu(z_0)\}$ clusters at a strongly pseudoconvex boundary point $\zeta_0 \in \partial \Omega$. In this situation, a theorem of B. Wong and J. P. Rosay shows that $\Omega$ is biholomorphically equivalent to $\mathbb{B}^n$, the unit ball in $\mathbb{C}^n$. Thus, while the hypotheses are purely local (the boundary $\partial \Omega$ being strongly pseudoconvex near $\zeta_0$), the conclusion is a global one, namely that $\Omega$ is biholomorphic to $\mathbb{B}^n$. There are several natural generalizations that explore this theme and they may be broadly categorized under three headings. These categories are obtained respectively by allowing notions that are weaker than strong pseudoconvexity on the boundary orbit accumulation point $\zeta_0$, by changing the ambient space in which $\Omega$ sits, and lastly replacing the holomorphic automorphisms $\{\phi_\nu\}$ by more general ‘analytic’ objects.

The first category has been explored in detail, starting with the program of Greene-Krantz to allow $\zeta_0$ to be a weakly pseudoconvex point. The theorems of F. Berteloot and Bedford-Pinchuk (cf. [1], [2]; for a comprehensive list of references, the reader is referred to the survey [11]) among others also belong to this class and in each case, a model domain is produced that is equivalent to the given domain $\Omega$. In the second category, two examples can be given. First, it is shown in [6] that the Wong-Rosay theorem holds when $\Omega$ is a domain in a complex manifold and the boundary orbit accumulation point $\zeta_0 \in \partial \Omega$ is strongly pseudoconvex. On the other hand, a version of this theorem (see [14] for a precise statement) holds even when $\Omega$ is a bounded, convex domain in a separable Hilbert space. Finally, it was shown in [25] that the Wong-Rosay theorem also holds even for a non-compact family of proper holomorphic self correspondences of $\Omega$, with uniformly bounded multiplicity, that admits a convex boundary orbit accumulation point of finite type. The strongly pseudoconvex case was studied in [19].

The principal aim of this article is to explore this phenomenon for a non-compact family of self isometries of a domain equipped with a Finsler metric such as the Kobayashi or the Carathéodory metric. By isometries we mean metric space isometries, which are, a priori, only continuous. The topology on the group of isometries of a space will be that of uniform convergence on compact subsets. Our main theorem is then the following.
Theorem 1.1. Let $\Omega$ be a bounded strongly convex domain in $\mathbb{C}^n$ with $C^6$-smooth boundary. Then the group of isometries of the Kobayashi distance (which equals the Carathéodory and the inner-Carathéodory distance) on $\Omega$ is compact, unless $\Omega$ is biholomorphic or anti-biholomorphic to the unit ball $B^n$. In particular, when $\Omega$ is equivalent to $B^n$, each isometry of $\Omega$ is biholomorphic or anti-biholomorphic.

A particularly useful strategy to study this phenomenon in the holomorphic category has been Pinchuk’s scaling method which leads to a quick proof of the original Wong-Rosay theorem. It may be recalled that scaling involves two steps. First, to show that the the given sequence of holomorphic automorphisms $\{\phi_\nu\}$ converge uniformly on compact sets of $\Omega$ to the constant mapping $\phi(z) \equiv \zeta_0, z \in \Omega$. And secondly, it needs to be shown that the composition of $\phi_\nu$ with the non-isotropic dilations is a normal family. The limit map will then yield the desired biholomorphic mapping between $\Omega$ and $B^n$. In trying to adapt these methods for a sequence of self-isometries of $\Omega$ with respect to some Finsler metric, three difficulties arise. First, it has to be proved that the given sequence of self-isometries will converge uniformly on compact sets of $\Omega$ to a constant mapping as in the holomorphic case (cf. Lemma 3.1). Second, a metric version of Pinchuk’s scaling method needs to be used to prove the ‘normality’ of the scaled isometries; this was developed in [24] for a different application and we intend to use it here as well. The conclusion then would be that the limit of the scaled isometries exists and is (only!) a continuous isometry between $\Omega$ and $B^n$ in the fixed Finsler metric.

The final and main difficulty (which is not seen in Pinchuk’s proof of the Wong-Rosay theorem) is to show that the continuous isometry is indeed a biholomorphic mapping. The proof of this is based on differential geometric considerations: In particular, the fact that the Kobayashi metric of the ball is a Kähler metric of constant holomorphic sectional curvature plays an important role. Our proof also requires the existence of complex geodesics and hence we restrict to convex domains.

The endeavour of showing that the continuous isometry is indeed a biholomorphic or anti-biholomorphic mapping can be compared with related results of authors such as I. Graham, G. Patrizio and J. P. Vigué among others (cf. [8], [9], [10], [20], [17], [27] and [28]. See also [3] for a similar result). A salient feature of these articles is the study of a holomorphic mapping $f$ between two domains in $\mathbb{C}^n$ that is an isometry at one point with respect to a given fixed Finsler metric. The conclusion then being that $f$ is actually biholomorphic. As noted in [13], these results can be traced back to an old conjecture of S. Kobayashi, a special case of which was noted by Krantz-Royden; they showed that a holomorphic self map of a strictly convex domain in $\mathbb{C}^n$ which is an isometry at a single point in the infinitesimal Kobayashi metric is indeed a biholomorphic mapping. In our situation, the assumption of having a global holomorphic mapping is replaced by a global isometry, the end conclusion remaining the same.

Finally, we believe that it should be possible, with some extra effort, to adapt our proof for dealing with strongly pseudoconvex domains. As mentioned above, the issue is that of showing that complex geodesics exist along “sufficiently” many directions.

2. Preliminaries

2.1. The Kobayashi and Carathéodory metrics. Let $\Delta$ denote the open unit disc in $\mathbb{C}$ and let $\rho(a, b)$ denote the distance between two points $a, b \in \Delta$ with respect to the Poïncare metric $F_\Delta$ (of constant curvature $-1$).

Let $\Omega$ be a domain in $\mathbb{C}^n$. The Kobayashi, Carathéodory and inner-Carathéodory distances on $\Omega$, denoted by $d_K^\Omega$, $d_C^\Omega$ and $d_{\tilde{C}}^\Omega$ respectively, are defined as follows:
Let \( z \in \Omega \) and \( v \in T_z \Omega \) a tangent vector at \( z \). Define the associated infinitesimal Kobayashi and Carathéodory metrics as

\[
F^K_\Omega(z,v) = \inf \left\{ \frac{1}{\alpha} : \alpha > 0, \text{ there exists } \phi \in \mathcal{O}(\Delta, \Omega) \text{ with } \phi(0) = z, \phi'(0) = \alpha v \right\}
\]
and

\[
F^C_\Omega(z,v) = \sup \{ df(z)v : f \in \mathcal{O}(\Omega, \Delta) \},
\]
respectively. The Kobayashi and the inner-Carathéodory length of a piece-wise \( C^1 \) curve \( \sigma : [0,1] \to \Omega \) are given by

\[
l^K(\sigma) = \int_0^1 F^K_\Omega(\sigma, \sigma') dt \quad \text{and} \quad l^C(\sigma) = \int_0^1 F^C_\Omega(\sigma, \sigma') dt
\]
respectively. Finally the Kobayashi and inner-Carathéodory distances between \( p, q \) are defined by

\[
d^K(p,q) = \inf l^K(p,q) \quad \text{and} \quad d^C(p,q) = \inf l^C(p,q),
\]
where the infimums are taken over all piece-wise \( C^1 \) curves in \( \Omega \) joining \( p \) and \( q \).

The Carathéodory distance is defined to be

\[
d^C(p,q) = \sup \{ \rho(f(p), f(q)) : f \in \mathcal{O}(\Omega, \Delta) \}.
\]

We note the following well-known and easy facts:

- If \( \Omega \) is a bounded domain, then \( d^K_\Omega, d^C_\Omega \) and \( F^K_\Omega \) are non-degenerate and the topology induced by these distances is the Euclidean topology.
- We always have

\[
d^C_\Omega(p,q) \leq d^C(p,q) \leq d^K_\Omega(p,q).
\]
- If \( \Omega = \mathbb{B}^n \), all the distance functions above coincide and are equal to the distance function of the Bergman metric \( g_0 \) on \( \mathbb{B}^n \). Here the Bergman metric is a complete Kähler metric normalized to have constant holomorphic sectional curvature \(-4\). Also, for \( \mathbb{B}^n \), the infinitesimal Kobayashi and Carathéodory metrics are both equal to the quadratic form associated to \( g_0 \).

### 2.2. Convexity and Pseudoconvexity

Suppose \( \Omega \) is a bounded domain in \( \mathbb{C}^n \), \( n \geq 2 \), with \( C^2 \)-smooth boundary. Let \( \rho : \mathbb{C}^n \to \mathbb{R} \) be a smooth defining function for \( \Omega \), i.e., \( \rho = 0 \) on \( \partial \Omega \), \( d\rho \neq 0 \) at any point of \( \partial \Omega \) and \( \rho^{-1}(\mathbb{R}_{>0}) = \overline{\Omega} \).

A domain with \( C^2 \) smooth boundary \( \Omega \) is said to be **strongly convex** if there is a defining function \( \rho \) for \( \partial \Omega \) such that the real Hessian of \( \rho \) is positive definite as a bilinear form on \( T_p \partial \Omega \), for every \( p \in \partial \Omega \).

\( \Omega \) is **strictly convex** if the interior of the straight line segment joining any two points in \( \overline{\Omega} \) is contained in \( \Omega \). Note that we do not demand the boundary of \( \Omega \) be smooth. Strong convexity implies strict convexity.

The **Levi form** of the defining function \( \rho \) at \( p \in \mathbb{C}^n \) is defined by

\[
L_p(v) = \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} (p)v_i \bar{v}_j \quad \text{for} \quad v = (v_1, \ldots, v_n) \in \mathbb{C}^n.
\]

For \( p \in \partial \Omega \), the maximal complex subspace of the tangent space \( T_p \partial \Omega \) is denoted by \( H_p(\partial \Omega) \) and called the **horizontal subspace** at \( p \). By definition, \( \Omega \) is **strongly pseudoconvex** if \( L_p \) is positive definite on \( H_p(\partial \Omega) \) for all \( p \in \partial \Omega \). It can be checked that strong convexity implies strong pseudoconvexity.

Let \( \Omega \) be a bounded domain. A holomorphic map \( \phi : \Delta \to \Omega \) is said to be an **extremal disc** or a complex geodesic for the Kobayashi metric (or distance) if it is distance preserving, i.e., \( d^K_\Omega(\phi(p), \phi(q)) = d_\Delta(p,q) \) for all \( p, q \in \Delta \).
Theorem 2.1. (L. Lempert [16]) Let $\Omega$ be a bounded strictly convex domain in $\mathbb{C}^n$.

(1) Given $p \in \Omega$ and $v \in \mathbb{C}^n$, there exists a complex geodesic $\phi$ with $\phi(0) = p$ and $\phi'(0) = v$ (or $d\phi(T_\Omega \Delta) = P_v$, where $P_v \subset T_p \Omega$ is the real two-plane associated to the complex vector $v$).

$\phi$ also preserves the infinitesimal metric, i.e., $F^K_\Omega(\phi(x); d\phi(w)) = F_\Delta(x; w)$ for all $x \in \Delta$ and $w \in T_x \Delta$.

(2) Given $p$ and $q$ in $\Omega$, there exists a complex geodesic $\phi$ whose image contains $p$ and $q$.

(3) The Kobayashi, Carathéodory and inner-Carathéodory distances coincide on $\Omega$. Also, the Kobayashi and Carathéodory infinitesimal metrics coincide on $\Omega$.

Theorem 2.2. (L. Lempert [16]) Let $\Omega$ be a bounded strongly convex domain in $\mathbb{C}^n$ with $C^k$ ($k \geq 6$) boundary. Then $F^K_\Omega : \Omega \times (\mathbb{C}^n \setminus \{0\}) \to \mathbb{R}$ is a $C^{k-5}$ smooth function.

2.3. Notation.

- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$, $\Delta_r := \{z \in \mathbb{C} : |z| < r\}$.
- $\rho_\Delta$ = distance function on $\Delta$ of the Poincaré metric of curvature $-4$.
- For $n \geq 2$, $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z| < 1\}$ and $B_a(r) = \{z \in \mathbb{C}^n : |z - a| < r\}$.
- $\Sigma = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : 2 \Re(z_n) + |z_1|^2 + \ldots + |z_{n-1}|^2 < 0\}$ = the unbounded realization of the ball $\mathbb{B}^n$ = Siegel domain.
- $d^K_{\Omega}$, $d^K_{\Omega}$ and $d^K_{\Omega}$ denote the Kobayashi, Carathéodory and inner-Carathéodory distances on $\Omega$.
- $F^K_\Omega$ and $F^K_{\Omega}$ denote the Kobayashi and Carathéodory infinitesimal metrics on $\Omega$.
- If $f : \Omega_1 \to \Omega_2$ is a smooth map between domains $\Omega_1$ and $\Omega_2$ in $\mathbb{C}^n$, then $d_f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denotes the derivative at $p \in \Omega_1$.

3. Compactness of the isometry group

The proof divides into two parts. In the first part, we get an isometry between $\Omega$ and $\mathbb{B}^n$ and in the second one we show that such an isometry has to be (anti)-holomorphic.

$\Omega$ IS ISOMETRIC TO $\mathbb{B}^n$:

For a bounded domain in $\mathbb{C}^n$, let $\text{Iso}^K(\Omega)$ and $\text{Iso}^C(\Omega)$ denote the group of $C^0$-isometries of $\Omega$ equipped with the Kobayashi and inner-Carathéodory distances. The topology on $\text{Iso}(\Omega)$ is that of uniform convergence on compact sets. We first note the following:

Lemma 3.1. Let $\Omega$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$. If $\text{Iso}^K(\Omega)$ is noncompact then there exists $p \in \partial \Omega$ and a sequence $\phi_n \in \text{Iso}^K(\Omega)$ with the following property: If $K \subset \Omega$ is any compact set, we have $\phi_n(x) \to p$ for any $x \in K$.

The same conclusion holds for $\text{Iso}^C(\Omega)$ also.

Proof. Let us first prove the existence of a point $z_0 \in \Omega$ and a sequence $\phi_n$ such that $\phi_n(z_0) \to \partial \Omega$. We suppose that such a pair does not exist and prove that every sequence $\phi_n \in \text{Iso}^K(\Omega)$ has a convergent subsequence, which would imply that $\text{Iso}^K(\Omega)$ is compact. It is clear that if $C \subset \Omega$ is any compact set then there is a compact set $S \subset \Omega$ such that $\phi_n(C) \subset S$ for all $n$. A normal families argument applied
to $\phi_\nu$ regarded as maps between the metric spaces $(\mathcal{C}, d^K_\Omega)$ and $(\mathcal{S}, d^K_\Omega)$ shows that $\phi_\nu$ has a convergent subsequence. Exhausting $\Omega$ by a sequence of compact sets and taking the diagonal subsequence completes the proof. We can assume that $\phi_\nu(z_0) \rightarrow p \in \partial \Omega$.

We now claim that for every open neighborhood $U$ of $p$ and every relatively compact subset $C$ of $\Omega$, it follows that $\phi_\nu(C) \subset \Omega \cap U$ for $\nu$ sufficiently large.

Indeed, fix a sufficiently small neighborhood $U$ of $p$, and a relatively compact subset $C$ of $\Omega$. Suppose that the aforementioned claim is false. Then there are points $z_\nu \in C$ such that $\phi_\nu(z_\nu) \in \Omega \setminus U$ for all $\nu$. Let $f : \Omega \rightarrow \Delta$ be a holomorphic peak function at $p$. Since $\phi_\nu$ are isometries, we have

$$d^K_{\Omega}(z_\nu) = d^K_\Omega(\phi_\nu(z_0), \phi_\nu(z_\nu))$$

for all $j$ and the distance decreasing property of the Kobayashi metric implies that

$$\rho_\Delta(f(\phi_\nu(z_0)), f(\phi_\nu(z_\nu))) \leq d^K_\Omega(\phi_\nu(z_0), \phi_\nu(z_\nu)).$$

The sequence $\{z_\nu\}$ is trapped in $C$ and hence the Kobayashi distance between $z_0$ and any of the $z_\nu$ is uniformly bounded above by $A > 0$. With this, the above equation implies

$$\frac{|f(\phi_\nu(z_0)) - f(\phi_\nu(z_\nu))|}{1 - f(\phi_\nu(z_0))f(\phi_\nu(z_\nu))} \leq \frac{\exp(2A) - 1}{\exp(2A) + 1} < 1.$$

But $f(\phi_\nu(z_0)) \rightarrow 1$ and if $\{\phi_\nu(z_\nu)\}$ clusters at $\zeta \in \Omega \setminus U$, then $|f(\zeta)| < 1$. Thus, the left side in the equation above approaches 1, which is a contradiction. $\square$

**Remark:** When working with the Kobayashi metric, the above reasoning does not require a global peak function. Indeed, a local peak function will suffice since the Kobayashi metric can be localized near such local peak points. The reader is referred to [6] where this localization is proved for domains in complex manifolds. Their ideas can be applied here as well to give a different argument for the same claim when $\Omega$ is not necessarily globally strongly pseudoconvex, but only locally so near $p$.

Let $p$ and $\phi_\nu$ be as above. We rescale as in [24] but only on ‘one side’, i.e. we fix a $z_0 \in \Omega$ (such that $\phi_\nu(z_0) \rightarrow p$) and rescale about the points $\phi_\nu(z_0)$ in conjunction with Lemma 3.1 as follows: let $q_\nu$ be the closest point on the boundary $\partial \Omega$ to $\phi_\nu(z_0)$. Such a point exists since $\partial \Omega$ is sufficiently smooth. Note that $q_\nu \rightarrow p$. Let $\rho(z)$ be the defining function for $\partial \Omega$. Then there exists a sequence of biholomorphic mappings $h_\nu : \mathbb{C}^n \rightarrow \mathbb{C}^n$ that map $q_\nu$ to the origin and such that the defining function for the domain $h_\nu(\Omega)$ near the origin takes the form

$$\rho \circ h^{-1}_\nu(z) = 2R(z_n + K_\nu(z)) + H_\nu(z) + \alpha_\nu(z)$$

where $K_\nu(z) = \sum_{i,j} a^\nu_{ij} z_i z_j, H_\nu(z) = \sum_{i,j} a^\nu_{ij} \bar{z}_i \bar{z}_j, \alpha(z) = o(|z|^2)$ with $K_\nu(\bar{z}, z) = 0$ and $H_\nu(\bar{z}, z) = |\bar{z}|^2$. Here $\bar{z}$ refers to the first $(n-1)$ coordinates of $z$, i.e., $\bar{z} = (z_1, \ldots, z_{n-1})$. Moreover, $h_\nu$ takes the real normal to $\partial \Omega$ at $q_\nu$ to the real normal $\{\bar{z} = y_n = 0\}$ to $\partial h_\nu(\Omega)$ at the origin. The existence of such mappings is well known (cf. [21]). Then $h_\nu(\phi_\nu(z_\nu)) = (0, -\delta_\nu)$ for some $\delta_\nu > 0$. Let $T_\nu : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the anisotropic dilatation given by

$$T_\nu(z, z_n) = (\bar{z}/\sqrt{\delta_\nu}, z_n/\delta_\nu).$$

Since each of $h_\nu, T_\nu$ are biholomorphic mappings, we get a sequence of scaled isometries

$$\Phi_\nu(z) : \Omega \rightarrow \Omega_\nu.$$
given by $\Phi_\nu(z) := T_\nu \circ h_\nu \circ \phi_\nu(z)$ with $\Omega_\nu := T_\nu \circ h_\nu \circ \phi_\nu(\Omega)$. The sequence of domains $\Omega_\nu$ converges to $\Sigma$, and the arguments used in Proposition 3.2 of [24] yield:

**Proposition 3.2.** If $C \subset \Sigma$ is a compact set, the Kobayashi distance and metric of $\Omega_\nu$ converge uniformly on $C$ to the Kobayashi distance and metric of $\Sigma$. Moreover there is a $C^0$-isometry $\Phi : \Omega \to \Sigma$, with respect to the Kobayashi metric on $\Omega$, such that the rescalings $\Phi_\nu$ converge uniformly to $\Phi$ on $C$. Similar statements hold with respect to the inner-Carathéodory metric if $\partial \Omega$ is assumed to be $C^3$ smooth.

**HOLOMORPHICITY OF THE ISOMETRY:**

From now on, we assume that $\Omega$ is a strongly convex domain. To complete the proof of Theorem 1.1, we will prove that a $C^0$-isometry between a strongly convex domain and the ball is holomorphic or anti-holomorphic. Since the Kobayashi and Carathéodory metrics coincide for strongly convex domains by Theorem 2.1, we drop the superscripts $K, C$.

**Lemma 3.3.** Let $\Omega$ be a strongly convex domain in $\mathbb{C}^n$ and $C$ a compact convex subset of $\Omega$. Then the Kobayashi distance of $\Omega$ and the Euclidean distance are Lipschitz equivalent on $C$.

**Proof.** By the compactness of $C$, there are constants $A, B$ such that

$$A|v| \leq F_\Omega(z, v) \leq B|v|$$

for all $z \in C$, $v \in \mathbb{C}^n$. Let $p, q \in C$. By considering the straight line segment joining $p, q$ and using (3.1), we see that $d_\Omega(p, q) \leq B|p - q|$ (note that the convexity of $C$ ensures that this line segment is in $C$). For the other inequality, let us note that the strong convexity of $\Omega$ guarantees (by Theorem 2.1) the existence of geodesics between any two points $p, q \in \Omega$. A geodesic in the metric space $(\Omega, d_\Omega)$ between $p, q$ is by definition an isometric map $\gamma : [0, d_\Omega(p, q)] \to \Omega$ with $\gamma(0) = p$ and $\gamma(d_\Omega(p, q)) = q$. Since $d_\Omega(p, \gamma(t)) \leq d_\Omega(p, q) \leq \text{diam}(C)$, we see that the image of $\gamma$ lies in the “tubular neighbourhood” $S := \{x \in \Omega : \inf_{y \in C} d_\Omega(x, y) \leq \text{diam}(C)\}$. Here the diameter is with respect to the intrinsic metric.

We note that $(\Omega, d_\Omega)$ is a complete metric space (cf. [7]). Hence $S$ is compact and we have an estimate similar to (3.1) for $S$. By using this estimate and the above fact that any geodesic connecting any $p, q \in C$ lies in $S$, we see that the length of such a geodesic is at least $c|p - q|$, for some constant $c$ which depends on $C$. \hfill $\Box$

**Proposition 3.4.** Let $\Omega$ be a strongly convex domain in $\mathbb{C}^n$ with $C^\infty$-smooth boundary. If $\Phi : \Omega \to \mathbb{B}^n$ is a $C^0$-isometry, then $\Phi$ is holomorphic or anti-holomorphic.

**Proof.** By Theorem 2.2, the Kobayashi (=Carathéodory) metric on $\Omega$ is $C^1$ smooth. Let us denote the Bergman metric on $\mathbb{B}^n$ by $g_0$. As noted in Section 2.2, this is a $C^\infty$ Kähler metric of constant holomorphic sectional curvature $-4$. The Kobayashi metric $F_{\mathbb{B}^n}$ of $\mathbb{B}^n$ is the quadratic form associated to $g_0$.

**Step 1:** $\Phi$ is differentiable almost everywhere.

By Lemma 3.3, the restriction of $\Phi$ to a convex domain $C$ with compact closure in $\Omega$ gives a Lipschitz map with respect to the Euclidean distance (the image of $C$ under $\Phi$ need not be convex. However it will be a subset of a relatively compact convex domain in $\mathbb{B}^n$, which is good enough for our purposes). Such a map is differentiable almost everywhere by the classical theorem of Rademacher-Stepanov. By exhausting $\Omega$ by a countable union of relatively compact convex domains, Step 1 is proved.

**Step 2:** $F_{\Omega}$ is Riemannian, i.e., $F_{\Omega}$ is the quadratic form associated to a Riemannian metric.
for all \(v,w \in \mathbb{R}^{2n}\). If we could ensure that \(F_{\Omega}\) is Riemannian on a dense subset of \(\Omega\), then by fixing \(v,w\) and using the continuity of \(F_{\Omega}\) in the domain variable, Step 2 would be proved.

In view of Step 1, it is hence enough to prove that \(F_{\Omega}\) is Riemannian at every point of differentiability of \(\Phi\). In fact we show that \(F_{\Omega} = \Phi^*(F_{\mathbb{B}^n})\) at every such point.

Let \(p\) be a point of differentiability of \(\Phi\) and let \(v \in T_p\Omega\). By Theorem 2.1, one can find a smooth geodesic (as defined in in the proof of Lemma 3.3) \(\gamma : [0, \epsilon) \to \Omega\) with \(\gamma(0) = p\) and \(\gamma'(0) = v\). Then \(\Phi \circ \gamma : [0,\epsilon) \to \mathbb{B}^n\) is a geodesic in a smooth Riemannian manifold and hence smooth (cf. [18]). Also

\[
l(\gamma|_{[0,t]}) = d_{\Omega}(p, \gamma(t)) = d_{\mathbb{B}^n}(\Phi(p), \Phi \circ \gamma(t)) = l(\Phi \circ \gamma|_{[0,t]}) \quad \text{for all } t \in [0,\epsilon),
\]

which implies that

\[
\int_0^t F_{\Omega}(\gamma(s), \gamma'(s))ds = \int_0^t F_{\mathbb{B}^n}((\Phi \circ \gamma)(s), (\Phi \circ \gamma)'(s))ds \quad \text{for all } t \in [0,\epsilon).
\]

Therefore

\[
F_{\mathbb{B}^n}((\Phi \circ \gamma)(s), (\Phi \circ \gamma)'(s)) = F_{\Omega}(\gamma(s), \gamma'(s)) \quad \text{for all } s \in [0,\epsilon).
\]

In particular, at \(s = 0\), we get \(F_{\Omega}(p,v) = F_{\mathbb{B}^n}(\Phi(p), d\Phi(v))\). Since this holds for any \(v\), we have \(F_{\Omega} = \Phi^*(F_{\mathbb{B}^n})\).

It should be noted that the Kobayashi metric is not, in general, Riemannian even for strongly convex domains [12].

**Step 3**: \(\Phi\) is \(C^1\) on \(\Omega\).

Since \(\Phi\) is a \(C^0\) isometry between the two \(C^1\) Riemannian manifolds \((\Omega, F_{\Omega})\) and \((\mathbb{B}^n, F_{\mathbb{B}^n})\), we can apply the classical theorem of Myers-Steenrod [18]:

Let \(M\) and \(N\) be smooth \(n\)-manifolds of class \(k+1\) and let \(g\) and \(h\) be \(C^k\) Riemannian metrics on \(M\) and \(N\). Then any \(C^0\) isometry \(\Phi : M \to N\) is a \(C^k\) smooth map and \(\Phi^*(h) = g\).

to conclude that \(\Phi\) is \(C^1\).

**Step 4**: \(\Phi\) is holomorphic/anti-holomorphic.

Let \(J_0\) and \(J\) denote the almost complex structures on \(T\mathbb{B}^n\) and \(T\Omega\) respectively. We want to prove that \(d\Phi \circ J = \pm J_0 \circ d\Phi\).

Fix \(p \in \Omega\). In order to prove that \(d\Phi : T_p\Omega \to T_{\Phi(p)}\mathbb{B}^n\) satisfies \(d\Phi \circ J = \pm J_0 \circ d\Phi\), we first prove that \(J\) invariant 2-planes go to \(J_0\) invariant 2-planes under \(d\Phi\). Let \(S_0\) and \(S\) denote the set of complex lines, i.e. 2-planes invariant under \(J_0\) and \(J\) respectively, in \(T_{\Phi(p)}\mathbb{B}^n\) and \(T_p\).

Since \(g_0\) has constant holomorphic sectional curvature \(-4\), at any point the sectional curvature \(Sec_{g_0}(P)\) of a 2-plane \(P\) spanned by an orthonormal pair of tangent vectors \(X, Y\) is \(Sec_{g_0}(P) = -1 + 3 < X, J_0Y >\). In particular, a two-dimensional subspace \(Q\) of \(T_{\Phi(p)}\mathbb{B}^n\) is in \(S_0\) if and only if \(Sec_{g_0}(Q) = -4\).

On the other hand, we claim that if \(P \in S\) then \(d\Phi(P) \in S_0\). This is seen as follows: given \(P \in S\), there exists a complex geodesic \(\phi : \Delta \to \Omega\) with \(\phi(0) = p\) and \(d\phi(T_0\Delta) = P\). The image \(\Phi \circ \phi(\Delta)\) is a \(C^\infty\) submanifold of \(\mathbb{B}^n\) since \(\Phi \circ \phi(\Delta) = exp_{g_0}(d\Phi(P))\) (this can be seen as follows: Since \(\Phi \circ \phi\) is distance preserving, it takes geodesics in \(\Delta\) to geodesics in \(\mathbb{B}^n\). Therefore \(\Phi \circ \phi(\Delta)\) is the union of geodesics which originate at \(\Phi(p)\) in directions along \(d\Phi(P)\). Since \(\Phi \circ \phi : \Delta \to \Phi \circ \phi(\Delta)\) is a distance-preserving map for the induced metric on the image, we can again invoke Myers-Steenrod to conclude that \(\Phi \circ \phi\) is at least a
$C^2$-isometry. This implies that the sectional curvature of $\Phi \circ \phi(\Delta)$ with respect to the induced metric is equal to that of $\Delta$ with respect to the hyperbolic metric, i.e. $-4$. On the other hand, $\Phi \circ \phi(\Delta)$ is a totally geodesic two-dimensional submanifold of $\mathbb{B}^n$ (because $\Phi \circ \phi$ is distance-preserving). Hence the sectional curvature of $\Phi \circ \phi(\Delta)$ at $\Phi(p)$ with respect to the induced metric is equal to the sectional curvature in the $g_0$ metric. Combining these two observations with the previous paragraph proves the claim.

Once we know that complex lines are taken to complex lines by $d\Phi$, we can use the fact that the metrics involved are invariant under the almost complex structures to conclude that $d\Phi \circ J = \pm J_0 \circ d\Phi$ on any $P \in S$. By the connectedness of $S$ as a subset of the Grassmann manifold of 2-planes in $T_p \Omega$, we see that $d\Phi \circ J = J_0 \circ d\Phi$ on every $P \in S$ or $d\Phi \circ J = -J_0 \circ d\Phi$ on every $P \in S$, i.e., $d\Phi \circ J = \pm J_0 \circ d\Phi$ on $T_p \Omega$. From the connectedness of $\Omega$ we conclude that $d\Phi \circ J = \pm J_0 \circ d\Phi$ on $T\Omega$. \hfill $\Box$

4. Appendix: Scaling of isometries

The purpose of this appendix is to show that $C^0$-isometries of a given domain can be scaled to yield a limit which is also a $C^0$-isometry, thereby verifying proposition 3.2. As mentioned in section 3, this was proved in [24] for isometries between in strongly pseudoconvex domains in $\mathbb{C}^n$ and to complement this result, we intend to focus on the following situation: $\mathcal{D}$ is a smoothly bounded, convex domain of finite type in $\mathbb{C}^n$ equipped with the Kobayashi metric and $\{\phi_\nu\}$ a sequence in $\text{Iso}^K(\mathcal{D})$ such that for some fixed $z_0 \in \mathcal{D}$, $\phi_\nu(z_0)$ converges to $\zeta_0$ in $\partial \mathcal{D}$. The goal will be to show that $\mathcal{D}$ is $C^0$-isometric to a convex model domain of finite type of the form $\tilde{\mathcal{D}} = \{z = (z_1, \tilde{z}) \in \mathbb{C} \times \mathbb{C}^{n-1} : \Im(z_1) + P(\tilde{z}) < 0\}$ where $P(\tilde{z})$ is a convex polynomial of degree at most $2m$ and the scaling used will be as in [4].

Fix a neighbourhood $U$ of $\zeta_0$ and $C$ a compact set in $\mathcal{D}$. Then $\phi_\nu(C) \subset U$ for $\nu$ large as the following argument shows. Suppose not. Then there are points $z_\nu \in C$ such that $\phi_\nu(z_\nu) \in \partial \mathcal{D}$ for all $\nu$. Since $\mathcal{D}$ is taut (each point on $\partial \mathcal{D}$ is a local plurisubharmonic peak point) and the Lempert function coincides with the Kobayashi and Carathéodory metric, it follows that there are holomorphic mappings $f_\nu : \Delta \to \mathcal{D}$ such that $f_\nu(0) = \phi_\nu(z_0)$, $f_\nu(\eta_\nu) = \phi_\nu(z_\nu)$ for some $\eta_\nu \in \Delta$ such that

$$\rho_\Delta(0, \eta_\nu) = d^K_\mathcal{D}(\phi_\nu(z_\nu), \phi_\nu(z_0)) = d^K_\mathcal{D}(z_\nu, z_0)$$

for all $\nu$. The Kobayashi distance between $z_0, z_\nu$ is uniformly bounded above for all $\nu$ since $z_\nu$ are all contained in $C$ and hence it follows that $|\eta_\nu| \leq 1 - \delta$ for some positive $\delta$. Let $\eta_\nu \to \eta$ with $|\eta| \leq 1 - \delta$ and let $f$ be the uniform limit on compact sets of the normal family $\{f_\nu\}$. Then $f(0) = \zeta_0$ and $f(\eta) = \zeta_0$, $f$ maps all of $\{\phi_\nu(z_\nu)\}$ to a cluster point of $\partial \mathcal{D}$. Thus there exists a non-trivial analytic disc $f$ whose centre passes through a local plurisubharmonic peak point and this is a contradiction.

For brevity write $q_\nu = \phi_\nu(z_\nu)$ and assume that $\zeta_0 = 0$ without loss of generality. Let $\mathcal{D} = \{r(z, \mathcal{Z}) < 0\}$ where $r(z, \mathcal{Z})$ is a smooth defining function for $\partial \mathcal{D}$ which has the form

$$r(z, \mathcal{Z}) = \Im(z_1) + \psi(\Re(z_1), z')$$

near the origin, with $\psi$ a smooth, convex function. Let $q^\nu_i \in \partial \mathcal{D}$ be closest to $q_\nu$. Denote the complex line containing $q_\nu, q^\nu_i$ by $l^\nu_i$ and let $\tau^\nu_i = |q_\nu - q^\nu_i|$. Since $\partial \mathcal{D}$ is of finite type, the distances from $q_\nu$ to $\partial \mathcal{D}$ in $(l^\nu_i)^\perp$ is uniformly bounded and there exists $q^\nu_i \in \partial \mathcal{D}$ where the largest distance is reached. Denote the complex line containing $q_\nu$ and $q^\nu_i$ by $l^\nu_i$ and set $\tau_i^\nu = |q_\nu - q^\nu_i|$. Now consider the orthogonal complement of the subspace generated by $l^\nu_i$ and $l^\nu_j$ and find the largest distance from $q_\nu$ to $\partial \mathcal{D}$ therein. Repeating this process we get orthogonal lines $l^\nu_1, l^\nu_2, \ldots, l^\nu_n$. Let $U_\nu$ be a unitary mapping of $\mathbb{C}^n$ sending $l^\nu_1$ to the $z_1$ axis and $q^\nu_i$ to a point on the positive imaginary axis $\Im(z_j)$. Then $T_\nu, T_\nu'$ be translations sending $q_\nu$ to the origin and the origin to $(-\tau_i^\nu, 0')$ respectively. The composition $h^\nu = T_\nu' \circ U_\nu \circ T_\nu$ gives a coordinate
system centered at $g_0^1$. Define the dilations

$$
\Lambda^\nu(z) = (z_1/\tau_1^\nu, z_2/\tau_2^\nu, \ldots, z_n/\tau_n^\nu)
$$

and the dilated domains

$$
\mathcal{D}_\nu = \{ z : r \circ (h^\nu)^{-1} \circ (\Lambda^\nu)^{-1}(z) < 0 \}.
$$

Note that $\mathcal{D}_\nu$ is convex and $(-i, 0') \in \mathcal{D}_\nu$ for all $\nu$. Among other things, the following two claims were proved in [4]. First, that $\mathcal{D}_\nu$ converges to

$$
\mathcal{G} = \{(z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1} : \Im(z_1) + P(z') < 0 \}
$$

a convex domain of finite type at most $2m$ and $P(z')$ is a convex polynomial of degree at most $2m$. Secondly, for all large $\nu$, $\mathcal{D}_\nu$ and hence $\mathcal{G}$ are contained in the intersection of half spaces $H_1 \cap H_2 \cap \cdots \cap H_n$, where

$$
H_1 = \{ z \in \mathbb{C}^n : \Im(z_1) < 0 \}
$$

and

$$
H_j = \{ z \in \mathbb{C}^n : \Im(\alpha_j z_j + \sum_{k<j} \alpha_{j,k} z_k) < 0 \}
$$

for $j \geq 2$ with $\alpha_j \in \mathbb{R} \setminus \{0\}$ for all $j \geq 2$ and $\alpha_{j,k} \in \mathbb{R}$ for all $j, k$. Note that $\mathcal{G}$ is complete hyperbolic in the sense of Kobayashi (cf. [5]).

The scaled maps $\Phi_\nu := \Lambda^\nu \circ h^\nu \circ \phi_\nu : \mathcal{D} \to \mathcal{D}_\nu$ are isometries in the Kobayashi metric on $\mathcal{D}$ and $\mathcal{D}_\nu$ and note that $\Phi_\nu(z_0) = (-i, 0')$ for all $\nu$. Exhaust $\mathcal{D}$ by an increasing union $\{S_i\}$ of relatively compact convex domains each containing $z_0$. Fix a pair $S_1 \subset \subset S_2$ say, and to show that $\{\Phi_\nu\}$ admits a convergent subsequence it will suffice to prove that $\{\Phi_\nu\}$ restricted to $S_1$ is uniformly bounded and equicontinuous. For $s_1, s_2 \in S_1$ note that

$$
d^K_{\mathcal{D}_\nu}(\Phi_\nu(s_1), \Phi_\nu(s_2)) = d^K_{\mathcal{D}}(s_1, s_2) \leq d^K_{S_1}(s_1, s_2) \leq C|s_1 - s_2|
$$

where the last inequality follows by the fact that in convex domains the Kobayashi distance is dominated by the standard distance. Moreover, $C$ is independent of $\nu$. Let $\pi_j : \mathbb{C}^n \to \mathbb{C}$ be the projection on the $j$-th coordinate axis and since $\mathcal{D}_\nu$ lies in the intersection of the half-planes $H_j$, it follows that $\pi_j(\mathcal{D}_\nu) \subset H_j$ for all $j \leq n$. Hence

$$
d^K_{H_j}(\pi_j(\Phi_\nu(s_1)), \pi_j(\Phi_\nu(s_2))) \leq d^K_{\mathcal{D}_\nu}(\Phi_\nu(s_1), \Phi_\nu(s_2))
$$

which when combined with (4.1) yields

$$
d^K_{H_j}(\pi_j(\Phi_\nu(s_1)), \pi_j(\Phi_\nu(s_2))) \leq C|s_1 - s_2|.
$$

To show that $\{\Phi_\nu(S_1)\}$ is uniformly bounded, let $s_2 = z_0$ and $s_1 \in S_1$ be arbitrary. Since the half-planes are all equivalent to the unit disc, it follows from (4.2) that each component of $\Phi_\nu$ is uniformly bounded and hence also that the images $\{\Phi_\nu(S_1)\}$ are contained uniformly relatively compactly in the intersection of all the half-planes. Equicontinuity of $\Phi_\nu$ restricted to $S_1$ is also a consequence of (4.2). Indeed, using the explicit form of the Kobayashi (which equals the Poincaré) metric on the half-planes, and the fact that the images are $\Phi_\nu(S_1)$ are relatively compactly contained in the intersection of the half-planes, it follows that

$$
|\pi_j(\Phi_\nu(s_1)) - \pi_j(\Phi_\nu(s_2))| \leq C|s_1 - s_2|
$$

for all $j \leq n$ where the constant $C$ can be chosen independent of $j$ and $s_1, s_2 \in S_1$. By Arzela-Ascoli, there is a well defined continuous limit of some subsequence of $\Phi_\nu$. Denote this limit by $\Phi : \mathcal{D} \to \overline{\mathcal{G}}$. If it were known that $\Phi$ is holomorphic, then the maximum principle would imply that $\Phi : \mathcal{D} \to \mathcal{G}$. However, $\Phi$ is known to be just continuous. To overcome this difficulty, let $\Omega \subset \mathcal{D}$ be the set of all points $z \in \mathcal{D}$ such
that $\Phi(z) \in \mathcal{G}$. $\Omega$ is non-empty since by construction $\Phi(z_0) = (-i, 0') \in \mathcal{G}$. Also, since $\Phi$ is continuous, it follows that $\Omega$ is open in $\mathcal{D}$.

Claim: It suffices to show that
\[
d^K_{\mathcal{D}}(p, q) = d^K_{\mathcal{G}}(\Phi(p), \Phi(q))
\]
for all $p, q \in \Omega$. Indeed, if $s \in \partial \Omega \cap \mathcal{D}$, choose a sequence $s_j \in \Omega$ that converges to $s$. If the claim were true, then
\[
d^K_{\mathcal{D}}(s_j, z_0) = d^K_{\mathcal{G}}(\Phi(s_j), \Phi(z_0))
\]
for all $j$ and since $s \in \partial \Omega \cap \mathcal{D}$, $\Phi(s_j) \to \partial \mathcal{G}$. As $\mathcal{G}$ is complete in the Kobayashi metric, the right side in (4.3) is unbounded, and this is a contradiction since the left side remains bounded. This would show that $\mathcal{D} = \Omega$. It is known that
\[
d^K_{\mathcal{D}}(p, q) = d^K_{\mathcal{D}_\nu}(\Phi_\nu(p), \Phi_\nu(q))
\]
for all $\nu$, and thus it remains to show that the right side above converges to $d^K_{\mathcal{G}}(\Phi(p), \Phi(q))$ as $\nu \to \infty$. For this note that
\[
|d^K_{\mathcal{D}_\nu}(\Phi_\nu(p), \Phi_\nu(q)) - d^K_{\mathcal{D}}(\Phi(p), \Phi(q))| \leq d^K_{\mathcal{D}_\nu}(\Phi_\nu(p), \Phi(p)) + d^K_{\mathcal{D}_\nu}(\Phi_\nu(q), \Phi(q))
\]
by the triangle inequality. Since $\Phi_\nu(p) \to \Phi(p)$ and the domains $\mathcal{D}_\nu$ converge to $\mathcal{G}$, it follows that there is a small ball $B(\Phi(p), r)$ around $\Phi(p)$ which contains $\Phi_\nu(p)$ for large $\nu$ and which is contained in $\mathcal{D}_\nu$ for all large $\nu$, where $r > 0$ is independent of $\nu$. Thus
\[
d^K_{\mathcal{D}_\nu}(\Phi_\nu(p), \Phi(p)) \leq C |\Phi_\nu(p) - \Phi(p)|
\]
for some uniform constant $C$. The same argument works for showing that $d^K_{\mathcal{D}}(\Phi_\nu(q), \Phi(q))$ is small. So to verify the claim, it is enough to prove that $d^K_{\mathcal{D}_\nu}(\Phi(p), \Phi(q))$ converges to $d^K_{\mathcal{G}}(\Phi(p), \Phi(q))$. To show this convergence, first note that by [30], it follows that
\[
F^K_{\mathcal{D}_\nu}(a, v) \to F^K_{\mathcal{G}}(a, v)
\]
for a fixed $a \in \mathcal{G}$ and a tangent vector $v$ at $a$. Moreover, the convergence is uniform on compact subsets of $\mathcal{G} \times \mathbb{C}^n$. To integrate this, let $\gamma : [0, 1] \to \mathcal{G}$ be a $C^1$ path with $\gamma(0) = \Phi(p)$, $\gamma(1) = \Phi(q)$ which almost realizes $d^K_{\mathcal{G}}(\Phi(p), \Phi(q))$, i.e.,
\[
\int_0^1 F^K_{\mathcal{G}}(\gamma, \gamma') \, dt \leq d^K_{\mathcal{G}}(\Phi(p), \Phi(q)) + \epsilon
\]
Since the trace of $\gamma$ is relatively compact in $\mathcal{G}$ and $\mathcal{D}_\nu \to \mathcal{G}$, it follows that the trace of $\gamma$ is contained uniformly relatively compactly in $\mathcal{D}_\nu$ for all large $\nu$. Now (4.4) shows that
\[
d^K_{\mathcal{D}_\nu}(\Phi(p), \Phi(q)) \leq \int_0^1 F^K_{\mathcal{D}_\nu}(\gamma, \gamma') \, dt \leq d^K_{\mathcal{G}}(\Phi(p), \Phi(q)) + 2\epsilon.
\]
Conversely, it follows by Lempert’s work that there exist $\eta_\nu > 1$ and holomorphic mappings $\tau_\nu : \Delta_{\eta_\nu} \to \mathcal{D}_\nu$ such that $\tau_\nu(0) = \Phi(p)$, $\tau_\nu(1) = \Phi(q)$ and
\[
\frac{1}{2} \log \left( \frac{\eta_\nu + 1}{\eta_\nu - 1} \right) = \rho_{\Delta_{\eta_\nu}}(0, 1) = d^K_{\mathcal{D}_\nu}(\Phi(p), \Phi(q)) = \int_0^1 F^K_{\mathcal{D}_\nu}(\tau_\nu(t), \tau_\nu'(t)) \, dt.
\]
By (4.5), $d^K_{\mathcal{D}_\nu}(\Phi(p), \Phi(q))$ is bounded above by $d^K_{\mathcal{G}}(\Phi(p), \Phi(q)) + 2\epsilon$ which is finite and hence it follows that $\eta_\nu > 1 + \delta$ for some uniform $\delta > 0$. Moreover, the domains $\mathcal{D}_\nu$ are all contained in the intersection
of the half-planes and hence the family $\tau_{\nu} : \Delta_{1+\delta} \to D_{\nu}$ is normal. Call the limit map $\tau : \Delta_{1+\delta} \to G$ and note that by construction $\tau(0) = \Phi(p)$ and $\tau(1) = \Phi(q)$. Since $\tau_{\nu} \to \tau$ uniformly on $[0, 1]$, it follows that

\begin{equation}
(4.6) \quad d^K_G(\Phi(p), \Phi(q)) \leq \int_0^1 F^K_G(\tau(t), \tau'(t))\, dt \leq \int_0^1 F^K_{\mathcal{D}_\nu}(\tau_{\nu}(t), \tau'_{\nu}(t))\, dt + \epsilon = d^K_{\mathcal{D}_\nu}(\Phi(p), \Phi(q)) + \epsilon.
\end{equation}

Combining this with (4.5) we conclude that the claim made above holds true and consequently that $D = \Omega$. Thus $\Phi : D \to G$ and that $d^K_G(p, q) = d^K_G(\Phi(p), \Phi(q))$ for all $p, q \in D$. It remains to show that $\Phi$ is surjective. For this, we consider the isometries $\Phi^{-1} : G \to D$. Exactly the same arguments as above show that there is a well defined uniform limit on compact sets of the sequence $\Phi^{-1}$, say $\Psi : G \to D$. Now $z = \Phi_0 \circ \Phi^{-1}(z) \to \Phi \circ \Psi(z)$ and likewise $\Psi \circ \Phi(z) = z$. It follows that $D$ is $C^0$-isometric to the model domain $G$.

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