EIGENVALUES OF KINEMATICAL CONSERVATION LAWS (KCL) BASED 3-D WEAKLY NONLINEAR RAY THEORY (WNLRT)

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ABSTRACT. Kinematical conservation laws (KCL) is a system of conservation laws governing the evolution of a curve in a plane or a surface in space, even if the curve or the surface has singularities on it. In our recent publication [1] we have developed a mathematical theory to study the successive positions and geometry of a 3-D weakly nonlinear wavefront by adding an energy transport equation to KCL. The 7×7 system of equations of this KCL based 3-D weakly nonlinear ray theory (WNLRT) is quite complex and explicit expressions for its two nonzero eigenvalues could not be obtained in [1]. In this short note, we use two different methods: (i) the equivalence of KCL and ray equations and (ii) the transformation of surface coordinates, to derive the same exact expressions for these eigenvalues. The explicit expressions for nonzero eigenvalues are important also for checking stability of any numerical scheme to solve 3-D WNLRT.

1. INTRODUCTION

In [1] we have developed a method to study the propagation of a surface Ω_t in three-dimensional space $\mathbb{R}^3$. The paper contains two systems of equations for an isotropic evolution of Ω_t . The first system, called the ray equations, is for the evolution of a smooth Ω_t and consists of equations¹

$$(1.1) \quad \frac{d\mathbf{x}}{dt} = m\mathbf{n}, \quad |\mathbf{n}| = 1,$$

$$(1.2) \quad \frac{d\mathbf{n}}{dt} = -\mathbf{L}m := -(\nabla - \mathbf{n}\langle \mathbf{n}, \nabla \rangle)m.$$

The second system, called 3-D KCL, consists of six conservation laws

$$(1.3) \quad (g_1\mathbf{u})_t - (m\mathbf{n})_{\xi_1} = 0,$$

$$(1.4) \quad (g_2\mathbf{v})_t - (m\mathbf{n})_{\xi_2} = 0.$$

The KCL (1.3)-(1.4) is to be supplemented by an additional condition $(g_2\mathbf{v})_{\xi_1} - (g_1\mathbf{u})_{\xi_2} = 0$ to be satisfied at time $t = 0$. Unlike the ray equations, the KCL remains valid also when Ω_t develops singularities.

As long as Ω_t remains smooth, the systems (1.1)-(1.2) and (1.3)-(1.4) are equivalent, we refer the reader to [1] for a proof of this basic result.

It is to be pointed out that both the systems (1.1)-(1.2) and (1.3)-(1.4) are under-determined. Hence, suitable closure relations are to be supplied to both to get a completely determined system of equations. In [1] we have derived a closure relation by considering the energy propagation along the rays of a weakly nonlinear ray theory (WNLRT) under high frequency approximation. For a

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¹For definition of various quantities and notations, see the reference [1], particularly Appendix B.

nonlinear wave front running into a polytropic gas at rest and in uniform state, the energy transport equation written in the form (7.4) of [1] is

$$(1.5) \quad \frac{\partial m}{\partial t} = \frac{m-1}{2n_3} \left(n_1 \frac{\partial}{\partial x_3} - n_3 \frac{\partial}{\partial x_1} \right) n_1 - \frac{m-1}{2n_3} \left(n_3 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_3} \right) n_2.$$

A conservation form of the equation (1.5) is the equation (6.7) of [1] which is

$$(1.6) \quad \left\{ (m-1)^2 e^{2(m-1)} g_1 g_2 \sin \chi \right\}_t = 0.$$

We add the differential form the energy transport equation, namely (1.5) to the ray equations (1.1)-(1.2). Henceforth, we shall designate the resulting system of equations as *ray equations based WNLRT*. Similarly, we will refer the KCL (1.3)-(1.4) along with the energy equation (1.6) in conservation form as *KCL based WNLRT*.

Since the two formulations of WNLRT are equivalent for smooth solutions, their nonzero eigenvalues must be same, see [1]-Section 8.4. The aim of the present work is to determine these eigenvalues of the KCL based WNLRT from the much simpler system of ray equations based WNLRT.

2. CALCULATION OF THE NONZERO EIGENVALUES

For the determination of eigenvalues, we do not need the first part of the ray equations, i.e., equation (1.1) but only the equation (1.2) for $\mathbf{n}$. However, since $|\mathbf{n}| = 1$, the vector equation (1.2) contains only two independent scalar equations. The two evolution equations for n_1 and n_2 are equations (5.9)-(5.10) of [1] which we reproduce below.

$$(2.1) \quad \frac{\partial n_1}{\partial t} = -\frac{1}{\sin^2 \chi} \left\{ (u_1 - v_1 \cos \chi) \frac{\partial m}{g_1 \partial \xi_1} + (v_1 - u_1 \cos \chi) \frac{\partial m}{g_2 \partial \xi_2} \right\},$$

$$(2.2) \quad \frac{\partial n_2}{\partial t} = -\frac{1}{\sin^2 \chi} \left\{ (u_2 - v_2 \cos \chi) \frac{\partial m}{g_1 \partial \xi_1} + (v_2 - u_2 \cos \chi) \frac{\partial m}{g_2 \partial \xi_2} \right\}.$$

Using the mapping between (x_1, x_2, x_3) -space and (ξ_1, ξ_2, t) -space, the expressions for the partial derivatives $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial x_3}$ can be derived in terms of $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \xi_1}$ and $\frac{\partial}{\partial \xi_2}$, see (5.6) of [1]. After the transformation of independent variables to (ξ_1, ξ_2, t) , the energy transport equation (1.5) becomes

$$(2.3) \quad \frac{\partial m}{\partial t} = \frac{m-1}{2n_3 \sin \chi} \left(v_2 \frac{\partial n_1}{g_1 \partial \xi_1} + v_1 \frac{\partial n_2}{g_1 \partial \xi_1} - u_2 \frac{\partial n_1}{g_2 \partial \xi_2} - u_1 \frac{\partial n_2}{g_2 \partial \xi_2} \right).$$

It is to be remarked that though the variables g_1 and g_2 do appear in equations (2.1)-(2.3), no derivatives of g_1 and g_2 are present. Hence, for the calculation of the nonzero eigenvalues, we need not consider the evolution equations for g_1 and g_2 given by the equations (6.10)-(6.11) of [1]. We can easily write down the characteristic equation of (2.1)-(2.3) and solve for the eigenvalues, which turn out to be

$$(2.4) \quad \lambda_1, \lambda_2 = (-\lambda_1), \lambda_3 = 0,$$

where

$$(2.5) \quad \lambda_1 = \left\{ \frac{m-1}{2 \sin^2 \chi} \left(\frac{e_1^2}{g_1^2} - \frac{2e_1 e_2}{g_1 g_2} \cos \chi + \frac{e_2^2}{g_2^2} \right) \right\}^{1/2}.$$

The explicit expressions for λ_1 and λ_2 in (2.4)-(2.5) are simple and elegant. They reduce to ν_1 and ν_2 in equation (8.2) of [1] when $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. The analytical expressions for nonzero eigenvalues of KCL are essential to determine a suitable CFL stability condition in the numerical integration of conservation laws of KCL based WNLRT, see [2] for more details.

3. CALCULATION OF NONZERO EIGENVALUES BY TRANSFORMATION OF SURFACE COORDINATES AT A POINT.

In this section we shall use the method of transformation of coordinates explained in section 8.3 of [1] to deduce the expression (2.5). Given the surface coordinates (ξ_1, ξ_2) at a point P_0 on Ω_t with unit vectors $(\mathbf{u}, \mathbf{v})$, we choose an orthogonal coordinates at P_0 with unit vectors $(\mathbf{u}', \mathbf{v}')$ as follows

$$(3.1) \quad \mathbf{u}' = \mathbf{u}, \quad \mathbf{v}' = \gamma_2 \mathbf{u} + \delta_2 \mathbf{v},$$

where $\langle \mathbf{u}', \mathbf{v}' \rangle = 0$ and $|\mathbf{v}'| = 1$ so that

$$(3.2) \quad \gamma_2 = -\cot \chi, \quad \delta_2 = \operatorname{cosec} \chi.$$

The sign of γ_2 and δ_2 are so chosen that vectors $\mathbf{u}', \mathbf{v}'$ and $\mathbf{n}'$ form a right handed system, similar to the three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{n}$. Substituting $\gamma_1 = 1$, $\delta_1 = 0$ and the above values of γ_2 and δ_2 in the expression (8.15) of [1] we get the same expression (2.5) for λ_1 .

4. CONCLUDING REMARKS

Though we could not derive the expression for λ_1 by solving the characteristic equation $\det(-\lambda A + e_1 B^{(1)} + e_2 B^{(2)}) = 0$ of the system (6.12) of [1], we have obtained its explicit expression (2.5) by two different methods. This leaves no doubt about the correctness of the expression in (2.5). However, just as a check, we have numerically evaluated the expression (2.5) and numerically solved the eigenvalue problem of the KCL based 3-D WNLRT with $g_1 = g_2 = 1$ and various values of $\mathbf{u}, \mathbf{v}$ and m . The numerical values of λ_1 agree in all cases, see also section (8.2) of [1].

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