THE ENTROPY OF HOLOMORPHIC CORRESPONDENCES: EXACT COMPUTATIONS AND RATIONAL SEMIGROUPS

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Abstract. We study two notions of topological entropy of correspondences introduced by Friedland and Dinh–Sibony. Upper bounds are known for both. We identify a class of holomorphic correspondences whose entropy in the sense of Dinh–Sibony equals the known upper bound. This provides an exact computation of the entropy for rational semigroups. We also explore a connection between these two notions of entropy.

1. Introduction, definitions and some results

This paper studies certain semigroups of holomorphic maps. It is motivated, however, by two related notions of topological entropy — one of which applies to meromorphic correspondences on a compact Kähler manifold, while the other applies, more generally, to closed relations on a compact metric space. Both notions are thus applicable to holomorphic correspondences (which we shall define presently) on a compact Kähler manifold. The first notion is due to Dinh and Sibony [3] while the second, introduced much earlier, is due to Friedland [4]. In both cases, upper bounds for each type of topological entropy were given: by Friedland in [4] and by Dinh–Sibony in [3]. However, for either type of entropy, this upper bound is in general **strictly greater** than the actual entropy. In this work, among other things, we identify a natural class of holomorphic correspondences for which this upper bound equals the entropy of Dinh–Sibony.

We use the word "natural" because the above-mentioned correspondences turn out to be correspondences representing certain semigroups of holomorphic maps. Hence, the main results in this paper will be stated for these semigroups. To get to these results, we need some definitions.

Definition 1.1. Let X_1 and X_2 be two compact, connected complex manifolds of dimension n. A *holomorphic correspondence* from X_1 to X_2 is a formal linear combination of the form

$$\Gamma = \sum\nolimits_{1\leq j\leq N} m_j \Gamma_j, \tag{1.1}$$

where the m_j's are positive integers and $\Gamma_1, \Gamma_2, \dots, \Gamma_N$ are distinct irreducible complex-analytic subvarieties of $X_1 \times X_2$ of pure dimension n that satisfy the following conditions:

(1) for each Γ_j in (1.1), $\pi_1|_{\Gamma_j}$ and $\pi_2|_{\Gamma_j}$ are surjective;

(2) for each $x \in X_1$ and $y \in X_2$, $\left(\pi_1^{-1}\{x\} \cap \Gamma_j\right)$ and $\left(\pi_2^{-1}\{y\} \cap \Gamma_j\right)$ are finite sets for each j;

where π_i is the projection onto X_i, $i = 1, 2$.

Given a holomorphic correspondence Γ from X_1 to X_2, the set (in terms of the notation in (1.1)) $\cup_{1\leq j\leq N}\Gamma_j$ is called the *support* of Γ, which we denote by $|\Gamma|$. The data $(m_1, \dots, m_N)$ in (1.1) are an essential part of the definition above. We shall elaborate on this below, but a brief reason is as follows. If $X_1 = X_2 = X$ in Definition 1.1, then we say that Γ is a *holomorphic correspondence* ***on*** X. Two holomorphic correspondences on X can be composed with each other. It is possible for a correspondence Γ, even if $m_1 = \dots = m_N = 1$, to be such that some of the irreducible components of $\Gamma \circ \Gamma$ occur with multiplicity higher than 1. The ability to compose two correspondences introduces the perspective of dynamics to the study of correspondences.

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We now introduce the two notions of entropy that we alluded to. We begin with the more general notion.

Definition 1.2 (Friedland, [4, 5])**.** Let X be a compact metric space and let Γ be a closed relation on X (i.e., Γ is a closed subset of $X \times X$ and the projection $\pi_1|_\Gamma$ is surjective). Let

$$X^{\mathbb{N}} := \{(x_0, x_1, x_2, \dots) : x_n \in X,\ n \in \mathbb{N}\}$$

endowed with the product topology, and let

$$\Gamma^\infty := \{(x_0, x_1, x_2, \dots) \in X^{\mathbb{N}} : (x_n, x_{n+1}) \in \Gamma\ \ \forall n \in \mathbb{N}\}.$$

If Γ^∞ is endowed with the topology that it inherits from $X^{\mathbb{N}}$, then, by definition, the left-shift on $X^{\mathbb{N}}$ induces a continuous map $\sigma : \Gamma^\infty \longrightarrow \Gamma^\infty$, where $\sigma : (x_0, x_1, x_2, \dots) \longmapsto (x_1, x_2, x_3, \dots)$. Then *Friedland's entropy* for Γ, denoted by $h_F(\Gamma)$, is defined as the topological entropy, in the sense of Bowen, of $\sigma : \Gamma^\infty \longrightarrow \Gamma^\infty$ (usually denoted by $h(\sigma)$ in the literature).

For the next definition, we need an alternative presentation of the correspondence introduced in Definition 1.1. We rewrite Γ as

$$\Gamma = \sum\nolimits'_{1 \leq j \leq M} \Gamma^\bullet_j, \tag{1.2}$$

where the primed sum indicates that the irreducible subvarieties $\Gamma^\bullet_j$, $j = 1, \dots, M$, are *not necessarily distinct* and are repeated according to multiplicity that is given by the coefficients $m_1, \dots, m_N$ in (1.1). Therefore, $M = m_1 + \dots + m_N$. With this explanation, we give

Definition 1.3 (Dinh–Sibony, [3])**.** Let X be compact, connected complex manifold and let Γ be a holomorphic correspondence on X. For each $\nu \in \mathbb{Z}_+$, a *ν-orbit of* Γ is any tuple of the form

$$(x_0, x_1, \dots, x_\nu; \alpha_1, \dots, \alpha_\nu) \in X^{\nu+1} \times \{1, \dots, M\}^\nu,$$

where $(x_{j-1}, x_j) \in \Gamma^\bullet_{\alpha_j}$, $j = 1, \dots, \nu$, assuming the presentation (1.2) for Γ. Fix a metric d compatible with the topology of X. If $\mathscr{F}$ is a family of ν-orbits, we say that $\mathscr{F}$ is an (ε, ν)-*separated family*, $\varepsilon > 0$, if for all pairs of distinct elements

$$(x_0, x_1, \dots, x_\nu; \alpha_1, \dots, \alpha_\nu) \quad \text{and} \quad (y_0, y_1, \dots, y_\nu; \beta_1, \dots, \beta_\nu)$$

of $\mathscr{F}$, we have

$$d(x_j, y_j) > \varepsilon \text{ for some } j = 0, 1, \dots, \nu \quad \text{or} \quad \alpha_j \neq \beta_j \quad \text{for some } j = 1, \dots, \nu. \tag{1.3}$$

Then, the *topological entropy of* Γ, denoted by $h_{top}(\Gamma)$, is defined as

$$h_{top}(\Gamma) := \sup_{\varepsilon > 0} \limsup_{\nu \to \infty} \frac{1}{\nu} \log\big(\max\{\sharp\mathscr{F} : \mathscr{F} \text{ is an } (\varepsilon, \nu)\text{-separated family}\}\big).$$

Remark 1.4. Since the manifold X is compact, for any $\varepsilon > 0$ and $\nu \in \mathbb{Z}_+$, any (ε, ν)-separated family in the above definition is finite. Furthermore, it is routine to verify (see [3, Section 4]) that $h_{top}(\Gamma)$ does not depend on the choice of the metric d for defining (ε, ν)-separatedness.

Definition 1.5. A *rational semigroup* is a semigroup, with composition of maps as the semigroup operation, whose elements are surjective holomorphic self-maps of $\mathbb{P}^n$ for some $n \in \mathbb{Z}_+$.

Remark 1.6. Note that, despite the word "rational" in Definition 1.5, the elements of a rational semigroup on $\mathbb{P}^n$, $n \geq 2$, do **not** possess indeterminacies. We have some results about classical rational semigroups (i.e., defined on $(\mathbb{C} \cup \{\infty\}) \cong \mathbb{P}^1$), and do not want to coin new terminology for theorems that also hold true for higher-dimensional analogues of the latter semigroups. This is the reason for the term "rational semigroups" introduced in Definition 1.5.

There is a very natural connection between *finitely* generated rational semigroups and holomorphic correspondences. Since this association defines the holomorphic correspondences for which we shall make exact entropy computations, let us state it formally. This association makes sense in greater generality, and not just for rational semigroups.

Definition 1.7. Let X be a compact, connected complex manifold and let S be a finitely generated semigroup consisting of surjective holomorphic self-maps of X. Let $\mathscr{G} = \{f_1, \dots, f_N\}$ be a set of generators of S. We call the holomorphic correspondence

$$\Gamma_{\mathscr{G}} := \sum\nolimits_{1 \leq j \leq N} \mathsf{graph}(f_j) \tag{1.4}$$

on X the *holomorphic correspondence associated with* $(S, \mathscr{G})$.

As has been observed earlier — see [7] by Ghys, Langevin and Walczak for the case of pseudogroups, or [5] — the entropy of a finitely-generated semigroup requires the specification of a set of generators. Thus, for X and S as in Definition 1.7, and for a choice $\mathscr{G}$ of a set of generators of S, we formally set:

$$h_F(S, \mathscr{G}) := h_F\big(|\Gamma_{\mathscr{G}}|\big) \quad \text{and} \quad h_{top}(S, \mathscr{G}) := h_{top}(\Gamma_{\mathscr{G}}).$$

Recall that if $f : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is holomorphic, then using homogeneous coordinates, we have:

$$f([z_0 : z_1 : \dots : z_n]) = \big[\mathsf{f}_0(z_0, z_1, \dots, z_n) : \mathsf{f}_1(z_0, z_1, \dots, z_n) : \dots : \mathsf{f}_n(z_0, z_1, \dots, z_n)\big],$$

where there exists a number $d_1(f) \in \mathbb{Z}_+$, and $\mathsf{f}_0, \mathsf{f}_1, \dots, \mathsf{f}_n$ are homogeneous polynomials of degree $d_1(f)$ such that $\cap_{i=0}^{n} \mathsf{f}_i^{-1}\{0\} = \{0\}$. With this definition, we can state our first result.

Theorem 1.8. *Let S be a finitely generated rational semigroup on $\mathbb{P}^n$ for some $n \in \mathbb{Z}_+$. Let $\mathscr{G} = \{f_1, \dots, f_N\}$ be a set of generators of S. Then*

$$h_{top}(S, \mathscr{G}) = \log\Big(\sum\nolimits_{1 \leq j \leq N} d_1(f_j)^n\Big).$$

We can say a lot more than Theorem 1.8. The latter is a consequence of a more **general** theorem, which provides bounds from above and below on $h_{top}(S, \mathscr{G})$ in a more general context — see Theorem 4.2 below. Here, however: as the notion of a rational semigroup first arose in the area of complex dynamics in one dimension (see [9] by Hinkkanen and Martin) — and to foreshadow Theorem 5.2 — we state the following special case:

Corollary 1.9. *Let S be a finitely generated rational semigroup on $\mathbb{P}^1$, and let $\mathscr{G} = \{f_1, \dots, f_N\}$ be a set of generators of S. Then*

$$h_{top}(S, \mathscr{G}) = \log\Big(\sum\nolimits_{1 \leq j \leq N} \deg(f_j)\Big).$$

We now turn to Friedland's entropy. Although it makes sense in a much more general setting, h_F turns out to be harder to compute. This is because, among other reasons, notions approximating concepts such as irreducible components, etc., are much less well-behaved outside the complex-analytic setting, and do not feature in Definition 1.2. In the complex-analytic setting, this leads to two difficulties that one can point to (with X here as in Definition 1.1):

- (i) A holomorphic correspondence Γ on X can be iterated. If $d_{top}(\Gamma)$ denotes the topological degree of Γ (see Section 2), then one has the identity $d_{top}(\Gamma^{\circ\nu}) = d_{top}(\Gamma)^\nu$ for any $\nu \in \mathbb{Z}_+$. The analogous identity for the ν-fold iterate of the **relation** $|\Gamma|$ — which is relevant to the entropy h_F — is not true in general. This vitiates computations of h_F.
- (ii) If X is Kähler, then either type of entropy is dominated by the quantity $\mathrm{lov}(\Gamma)$ (see Section 2) — which results from a technique of Gromov [8]. For similar reasons as in (i), $\mathrm{lov}(\Gamma)$ turns out not to be the best upper bound for $h_F(\Gamma)$ even for $\Gamma = \Gamma_{\mathscr{G}}$.

Rather few examples of **exact** computations of h_F (not necessarily in the holomorphic category) are known: see, for instance, [5, Section 5] and [6] by Geller and Pollicott. But, as indicated above, computing h_F is inherently hard. However, certain lower bounds for $h_F(S, \mathscr{G})$, $\mathscr{G}$ finite, are almost immediate: $h_F(S, \mathscr{G}) \geq h_F(\langle f : f \in \mathscr{G}' \rangle, \mathscr{G}')$ for any $\varnothing \neq \mathscr{G}' \subsetneq \mathscr{G}$ (see Section 5 for details). In contrast to this, for S a rational semigroup on $\mathbb{P}^1$, we shall establish a lower bound for $h_F(S, \mathscr{G})$ that takes into consideration each of the generators in $\mathscr{G}$. This is our Theorem 5.2.

Since it requires some notation, we present it in Section 5. This theorem, in turn, relies on our central proposition of Section 3, which might be of independent interest. The proof of Theorem 1.8 (from which Corollary 1.9 is immediate) is presented in Section 4.

2. Complex geometry preliminaries

This section is devoted to a discussion of terminology from geometry appearing in Section 1 whose definitions had been deferred, and to stating a result that constitutes one part of the proofs of Theorem 1.8 and Corollary 1.9.

We first begin with a discussion of the composition of two holomorphic correspondences. Since one needs to understand this **only** to define a certain finite sequence of numbers associated to a correspondence Γ, we shall be brief. We refer the reader to [3, Section 3] for details (with a note to those unfamiliar with holomorphic correspondences that the footnote to [3, Section 3] is irrelevant in the case of holomorphic correspondences). We focus on two points that are relevant to this article (in what follows, $\Gamma^{\circ\nu}$ will denote the ν^{th} iterated composition of Γ):

(i) With X as in Definition 1.1, consider two holomorphic correspondences

$$\Gamma^1 \;=\; {\sum}'_{1\leq j\leq M_1}\Gamma^{\bullet}_{1,\,j} \qquad \text{and} \qquad \Gamma^2 \;=\; {\sum}'_{1\leq k\leq M_2}\Gamma^{\bullet}_{2,\,k}$$

on X, written in accordance with the presentation (1.2). The support of $\Gamma^2\circ\Gamma^1$ is just the classical composition of $|\Gamma^2|$ with $|\Gamma^1|$ as **relations**. Let us denote the latter composition by $\star$. If $Y_{s,\,jk}$, $s=1,\dots,M(j,k)$, are the distinct irreducible components of $|\Gamma^{\bullet}_{2,\,k}|\star|\Gamma^{\bullet}_{1,\,j}|$, then let

$$\begin{aligned}\eta_{s,\,jk} \;:=\;&\text{the number of } y\text{'s, for a generic } (x,z)\in Y_{s,\,jk}, \text{ such that}\\ &(x,y)\in\Gamma^{\bullet}_{1,\,j}\ \&\ (y,z)\in\Gamma^{\bullet}_{2,\,k}.\end{aligned}$$

Then, the definition in [3, Section 3] results in the formula:

$$\Gamma^2\circ\Gamma^1 \;:=\; \sum_{1\leq j\leq M_1}\ \sum_{1\leq k\leq M_2}\ \sum_{1\leq s\leq M(j,\,k)} \eta_{s,\,jk}Y_{s,\,jk}.$$

(ii) For the semigroup S, a choice of a set of generators $\mathscr{G}$, and the correspondence $\Gamma_{\mathscr{G}}$ introduced in Definition 1.7, we have

$$\Gamma^{\circ 2}_{\mathscr{G}} \;:=\; \Gamma_{\mathscr{G}}\circ\Gamma_{\mathscr{G}} \;=\; \sum\nolimits_{1\leq j,\,k\leq N}\mathsf{graph}(g_j\circ g_k).$$

Observe that if S is not a free semigroup and if, for instance, there exists a relation of the form $g_{j_1}\circ g_{k_1} = g_{j_2}\circ g_{k_2}$ for $(j_1,k_1)\neq(j_2,k_2)$, then the irreducible variety $\mathsf{graph}(g_{j_1}\circ g_{k_1})$ would occur with multiplicity at least 2. Observations such as the one above are the reason why the data $(m_1,\dots,m_N)$ in (1.1) are essential in defining a holomorphic correspondence.

One can pull back certain types of currents by a holomorphic correspondence — see [2, Section 3.1]. The *formal* prescription for the pullback (which we denote by F^*_{Γ}) of any current T of bidegree (p,p), $p=0,1,\dots,n$ (recall that $\dim_{\mathbb{C}}(X)=n$) is:

$$F^*_{\Gamma}(T) \;:=\; (\pi_1)_*\left(\pi_2^*(T)\wedge[\Gamma]\right)$$

whenever the intersection $\pi_2^*(T)\wedge[\Gamma]$ makes sense. Here, Γ detemines a current of bidimension (n,n) given by the currents of integration defined by its constituent subvarieties — which we denote by $[\Gamma]$. For example: any smooth (p,p)-form Θ on X, $p=0,1,\dots,n$, can be pulled back by Γ to give a (p,p)-current (equivalenty, a current of bidimension $(n-p,n-p)$) as follows:

$$\langle F^*_{\Gamma}(\Theta),\,\varphi\rangle \;:=\; \sum_{j=1}^{N} m_j\int_{\mathsf{reg}(\Gamma_j)}\big(\,\pi_2|_{\Gamma_j}\big)^*\Theta\wedge\big(\,\pi_1|_{\Gamma_j}\big)^*\varphi \quad \forall(n-p,n-p)\text{-forms }\varphi,$$

using the presentation (1.1) for Γ. Now suppose (X, ω) is a Kähler manifold and let ω_X denote the normalisation of ω so that $\int_X \omega_X^n = 1$. For $p = 0, 1, \dots, n$, we define the p^{th} *intermediate degree of* Γ by

$$\lambda_p(\Gamma) := \langle F_\Gamma^*(\omega_X^p), \omega_X^{n-p} \rangle.$$

It is well known that for each p, λ_p is sub-multiplicative with respect to composition. Thus, the limit on the right-hand side below

$$d_p(\Gamma) := \lim_{\nu \to \infty} \lambda_p(\Gamma^{\circ \nu})^{1/\nu}, \quad p = 0, 1, \dots, n, \tag{2.1}$$

exists. The number $d_p(\Gamma)$ is called the p^{th} *dynamical degree of* Γ. Since the limit on the right-hand side of (2.1) exists, $d_p(\Gamma^{\circ k}) = d_p(\Gamma)^k$, $p = 0, 1, \dots, n$, for every $k \in \mathbb{Z}_+$.

With these definitions, we can state a result that we shall need in proving Theorem 1.8 and Corollary 1.9.

Result 2.1 (paraphrasing of [3, Theorem 1.1])**.** *Let* (X, ω) *be a compact Kähler manifold of dimension* n *and let* Γ *be a holomorphic correspondence on* X*. Then*

$$h_{top}(\Gamma) \leq \max_{0 \leq p \leq n} \log d_p(\Gamma).$$

We ought to mention that Dinh–Sibony establish the above bound on h_{top} for the more general class of *meromorphic* correspondences. Furthermore, this bound is actually obtained — adapting a technique of Gromov [8] — by computing the value of $\text{lov}(\Gamma)$, which dominates $h_{top}(\Gamma)$. Roughly speaking, $\text{lov}(\Gamma)$ is the asymptotic rate of logarithmic growth (relative to ν) of the volume of the space of all ν-orbits.

3. NOTATION AND ESSENTIAL PROPOSITIONS ON TOPOLOGICAL ENTROPY

We begin by fixing some notation that will be needed for the propositions in this section and in subsequent sections. The objects introduced here will pertain to a *general* holomorphic correspondence Γ, and our notation will be with reference to the presentation (1.2) of Γ.

We begin by introducing an object similar to Γ^∞ of Definition 1.2. The parameter M has the same meaning in the following definition as in (1.2):

$${}^{\Gamma}X := \{(x_0, x_1, x_2, \dots; \alpha_1, \alpha_2, \dots) \in X^{\mathbb{N}} \times \{1, \dots, M\}^{\mathbb{Z}_+} : (x_{\nu-1}, x_\nu) \in \Gamma_{\alpha_\nu}^\bullet \ \ \forall \nu \in \mathbb{Z}_+\}.$$

This space is endowed with the topology that it inherits from $X^{\mathbb{N}} \times \{1, \dots, M\}^{\mathbb{Z}_+}$ endowed with the product topology. We will denote by ${}^{\Gamma}\mathscr{O}_\nu$ the space of all ν-orbits, i.e.,

$${}^{\Gamma}\mathscr{O}_\nu := \{(x_0, \dots x_\nu; \alpha_1, \dots, \alpha_\nu) \in X^{\nu+1} \times \{1, \dots, M\}^\nu : (x_{j-1}, x_j) \in \Gamma_{\alpha_j}^\bullet, \ \ 1 \leq j \leq \nu\}.$$

The above is endowed with the relative topology that it inherits from $X^{\nu+1} \times \{1, \dots, M\}^\nu$.

We shall need the following maps. By a mild abuse of notation, we shall denote by π_{itn} either the map $\pi_{\text{itn}} : {}^{\Gamma}X \longrightarrow X^{\mathbb{N}}$ or the map $\pi_{\text{itn}} : {}^{\Gamma}\mathscr{O}_\nu \longrightarrow X^{\nu+1}$ that maps the relevant orbit of an iteration under Γ to the itinerary of points in X along that orbit. In other words:

$$\pi_{\text{itn}} : {}^{\Gamma}X \ni (x_0, x_1, \dots; \alpha_1, \dots) \mapsto (x_0, x_1, \dots) \ \text{ or}$$

$$\pi_{\text{itn}} : {}^{\Gamma}\mathscr{O}_\nu \ni (x_0, x_1, \dots, x_\nu; \alpha_1, \dots, \alpha_\nu) \mapsto (x_0, x_1, \dots, x_\nu) \ \text{ respectively,}$$

where the precise definition of π_{itn} will be **obvious** from the context. By a similar abuse of notation, we shall denote by π_{symb} either the map $\pi_{\text{symb}} : {}^{\Gamma}X \longrightarrow \{1, \dots, M\}^{\mathbb{Z}_+}$ or the map $\pi_{\text{symb}} : {}^{\Gamma}\mathscr{O}_\nu \longrightarrow \{1, \dots, M\}^\nu$, defined by

$$\pi_{\text{symb}} : {}^{\Gamma}X \ni (x_0, x_1, \dots; \alpha_1, \dots) \mapsto (\alpha_1, \dots) \quad \text{or}$$

$$\pi_{\text{symb}} : {}^{\Gamma}\mathscr{O}_\nu \ni (x_0, x_1, \dots, x_\nu; \alpha_1, \dots, \alpha_\nu) \mapsto (\alpha_1, \dots, \alpha_\nu) \ \text{ respectively.}$$

Lastly, $\mathsf{start} : {}^{\Gamma}\mathscr{O}_\nu \longrightarrow X$ will denote the map $(x_0, x_1, \dots, x_\nu; \alpha_1, \dots, \alpha_\nu) \mapsto x_0$.

We shall also need a standard result in elementary topology (we abbreviate $\sigma^{\circ j}$ as σ^j):

Lemma 3.1. *Let* (Y, D) *be a compact metric space and let* $\widehat{D}$ *denote the metric*

$$\widehat{D}(\widehat{x}, \widehat{y}) := \sup_{n \in \mathbb{N}} \frac{D(x_n, y_n)}{2^n},$$

$\widehat{x} := (x_0, x_1, x_2, \dots)$ *and* $\widehat{y} := (y_0, y_1, y_2, \dots)$, *which metrises the product topology on* $Y^{\mathbb{N}}$. *Let* $\sigma : (x_0, x_1, x_2, \dots) \longmapsto (x_1, x_2, x_3, \dots)$ *be the left-shift on* $Y^{\mathbb{N}}$. *Then:*

$$\max_{0 \le j \le n} \widehat{D}\big(\sigma^j(\widehat{x}), \sigma^j(\widehat{y})\big) = \sup_{j \in \mathbb{N}} \frac{D(x_j, y_j)}{2^{(j-n)+}},$$

where $(j - n)+ := \max(j - n, 0)$.

Before stating the principal result of this section, we make a clarification. If Y is a compact metric space, and $f : Y \longrightarrow Y$ is a continuous map, then the *topological entropy of* f, i.e., of a **map** — which we shall denote by $h(f)$ — will mean the entropy introduced by Bowen [1]. (Bowen's definition does not require Y to be compact. In that case, f must be uniformly continuous relative to the metric on Y, and the value of Bowen's entropy of f depends on this metric. But when Y is compact, then Bowen's entropy is independent of the metric, provided it metrises the topology on Y.)

We now state and prove a result that will be needed in the proof of Theorem 5.2. This result is hinted at in [3, Section 4]. However:

- It is unclear if Proposition 3.2 follows, as alluded to in [3], from the conjugacy invariance of topological entropy (which applies to pairs of maps).
- For Γ as in Definition 1.1, if the topological degree of $\big(\pi_1|_{\Gamma_j}\big) \ge 2$ for any $j \in \{1, \dots, N\}$, then it is unclear whether Γ can at all be conjugated to a shift on ${}^{\Gamma}X$.

While we shall apply Proposition 3.2 only to the holomorphic correspondence $\Gamma_{\mathscr{G}}$ in Section 5, it holds true for general holomorphic correspondences. It may thus be of independent interest. In view of the two points above, it seems worthwhile to state and give a *direct* proof of

Proposition 3.2. *Let* X *and* Γ *be as in Definition 1.1. For* $\varepsilon > 0$ *and* $\nu \in \mathbb{Z}_+$, *let*

$N(\varepsilon, \nu)$:= *the cardinality of any* (ε, ν)*-separated family of* ν*-orbits, in the sense of Definition 1.3, having the greatest possible cardinality.*

Let $\mathscr{S}$ *denote the restriction of the shift map*

$$\sigma : (x_0, x_1, x_2, \dots; \alpha_1, \alpha_2, \dots) \mapsto (x_1, x_2, x_3, \dots; \alpha_2, \alpha_3, \dots)$$

to ${}^{\Gamma}X$. *Then,*

$$h_{top}(\Gamma) := \sup_{\varepsilon > 0} \limsup_{\nu \to \infty} \frac{1}{\nu} \log N(\varepsilon, \nu) = h(\mathscr{S}),$$

where $h(\mathscr{S})$ *is the entropy, in the sense of Bowen, of the continuous map* $\mathscr{S} : {}^{\Gamma}X \longrightarrow {}^{\Gamma}X$.

Proof. Let us fix a metric d on the complex manifold X that is compatible with the manifold topology. We choose the metric

$$\Delta\big((x_0, x_1, \dots; \alpha_1, \dots), (y_0, y_1, \dots; \beta_1, \dots)\big) := \max\left[\sup_{\nu \in \mathbb{N}} \frac{d(x_\nu, y_\nu)}{2^\nu}, \ \sup_{\nu \in \mathbb{N}} \frac{\boldsymbol{\delta}(\alpha_{\nu+1}, \beta_{\nu+1})}{2^\nu}\right]$$

(where $\boldsymbol{\delta}$ denotes the 0-1 metric on the symbols $\{1, \dots, M\}$) which metrises the topology on ${}^{\Gamma}X$. Since we must show that $h_{top}(\Gamma)$ equals the Bowen entropy of $\mathscr{S}$, we introduce, for $\varepsilon > 0$ and $\nu \in \mathbb{Z}_+$, the set

$M(\varepsilon, \nu)$:= the cardinality of any ε-separated set of orbits of $\mathscr{S}$ of duration ν having the greatest possible cardinality.

Recall that any two orbits,

$$\mathscr{O}_1 := (x_0, x_1, \dots; \alpha_1, \dots) \quad \text{and} \quad \mathscr{O}_2 := (y_0, y_1, \dots; \beta_1, \dots), \tag{3.1}$$

belonging to any of the sets referred to in the definition of $M(\varepsilon, \nu)$ satisfy

$$\max_{0 \le j \le \nu} \Delta\big(\mathscr{S}^j(\mathscr{O}_1), \mathscr{S}^j(\mathscr{O}_2)\big) > \varepsilon. \tag{3.2}$$

Fix a $\nu \in \mathbb{Z}_+$. It suffices to consider $\varepsilon \in (0,1)$.

Let $S(\varepsilon, \nu) \subset {}^{\Gamma}\mathscr{O}_\nu$ be an (ε, ν)-separated family, in the sense of Definition 1.3, such that $\sharp S(\varepsilon, \nu) = N(\varepsilon, \nu)$. For each ν-orbit $\mathsf{x} \in S(\varepsilon, \nu)$, let us pick a $(x_0, x_1, x_2, \dots; \alpha_1, \alpha_2 \dots) \in {}^{\Gamma}X$ such that

$$(x_0, x_1, \dots, x_\nu; \alpha_1, \dots, \alpha_\nu) = \mathsf{x},$$

and **fix** it. Call the latter infinite orbit $\widetilde{\mathsf{x}}$. Let us consider two distinct ν-orbits

$$\mathsf{x} := (x_0, x_1, \dots, x_\nu; \alpha_1, \dots, \alpha_\nu) \quad \text{and} \quad \mathsf{y} := (y_0, y_1, \dots, y_\nu; \beta_1, \dots, \beta_\nu)$$

belonging to $S(\varepsilon, \nu)$. We have two possibilities for the pair $\{\mathsf{x}, \mathsf{y}\}$:

Case 1. $\max_{0 \le j \le \nu} d(x_j, y_j) > \varepsilon$.
Then (with the meaning of $\widetilde{\mathsf{y}}$ hopefully being clear) by the definition of Δ, and in view of Lemma 3.1, we have

$$\max_{0 \le j \le \nu} \Delta\big(\mathscr{S}^j(\widetilde{\mathsf{x}}), \mathscr{S}^j(\widetilde{\mathsf{y}})\big) \ge \sup_{j \in \mathbb{N}} \frac{d(x_j, y_j)}{2^{(j-\nu)+}} > \varepsilon. \tag{3.3}$$

Case 2. $\max_{0 \le j \le \nu} d(x_j, y_j) \le \varepsilon$.
In this case, by (1.3) there exists a j^*, with $1 \le j^* \le \nu$, such that $\alpha_{j^*} \ne \beta_{j^*}$. Therefore, in view of Lemma 3.1, we have

$$\max_{0 \le j \le \nu} \Delta\big(\mathscr{S}^j(\widetilde{\mathsf{x}}), \mathscr{S}^j(\widetilde{\mathsf{y}})\big) \ge \sup_{j \in \mathbb{N}} \frac{\boldsymbol{\delta}(\alpha_{j+1}, \beta_{j+1})}{2^{(j-\nu)+}} \ge 1 > \varepsilon. \tag{3.4}$$

From (3.3) and (3.4) it follows that the set $\{\widetilde{\mathsf{x}} \in {}^{\Gamma}\mathscr{O} : \mathsf{x} \in S(\varepsilon, \nu)\}$ is an ε-separated set of orbits of $\mathscr{S}$ in the sense of (3.2). Since the latter set has cardinality $N(\varepsilon, \nu)$, we get:

$$M(\varepsilon, \nu) \ge N(\varepsilon, \nu). \tag{3.5}$$

Now let $\Sigma(\varepsilon, \nu) \subset {}^{\Gamma}X$ be an ε-separated set of orbits in the sense of (3.2) such that $\sharp\Sigma(\varepsilon, \nu) = M(\varepsilon, \nu)$. Then, for two distinct orbits $\mathscr{O}_1, \mathscr{O}_2 \in \Sigma(\varepsilon, \nu)$, we have (using the notation in (3.1)):

$$\frac{d(x_j, y_j)}{2^{(j-\nu)+}} \le \varepsilon \quad \forall j \ge \nu + \log_2(1/\varepsilon) + \log_2(\operatorname{diam}(X)), \tag{3.6}$$

$$\frac{\boldsymbol{\delta}(\alpha_{j+1}, \beta_{j+1})}{2^{(j-\nu)+}} \le \varepsilon \quad \forall j \ge \nu + \log_2(1/\varepsilon), \tag{3.7}$$

where $\log_2(t) := \log(t)/\log(2)\ \forall t > 0$. Given the definition of the metric Δ, it is impossible for the quantities

$$\max_{0 \le j \le \nu} \sup_{k \in \mathbb{N}} \frac{d\big(\pi_k \circ \pi_{\rm itn}(\mathscr{S}^j(\mathscr{O}_1)), \pi_k \circ \pi_{\rm itn}(\mathscr{S}^j(\mathscr{O}_2))\big)}{2^k},$$

and

$$\max_{0 \le j \le \nu} \sup_{k \in \mathbb{Z}_+} \frac{\boldsymbol{\delta}\big(\pi_k \circ \pi_{\rm symb}(\mathscr{S}^j(\mathscr{O}_1)), \pi_k \circ \pi_{\rm symb}(\mathscr{S}^j(\mathscr{O}_2))\big)}{2^{k-1}}$$

(where π_k denotes the projection onto the k^{th} factor) to **both** be less than or equal to ε. Thus, by (3.6), (3.7) and Lemma 3.1, we have

$$d(x_j, y_j) > 2^{(j-\nu)+}\varepsilon \quad \text{or} \quad \boldsymbol{\delta}(\alpha_{j+1}, \beta_{j+1}) \ne 0 \quad \text{for some } j : 0 \le j \le C(\varepsilon) + \nu,$$

where $C(\varepsilon)$ is the greatest integer that is strictly less than $\log_2(1/\varepsilon) + \log_2(\mathrm{diam}(X))$. Hence

$$(x_0, x_1 \dots, x_{C(\varepsilon)+\nu}; \alpha_1, \dots, \alpha_{C(\varepsilon)+\nu}) \quad \text{and} \quad (y_0, y_1 \dots, y_{C(\varepsilon)+\nu}; \beta_1, \dots, \beta_{C(\varepsilon)+\nu})$$

are $(\varepsilon, C(\varepsilon)+\nu)$-separated in the sense of Definition 1.3. Since this applies to any pair of distinct $\mathscr{O}_1, \mathscr{O}_2 \in \Sigma(\varepsilon, \nu)$, we get

$$N(\varepsilon, C(\varepsilon) + \nu) \geq M(\varepsilon, \nu).$$

From this and (3.5), it follows that

$$N(\varepsilon, \nu) \leq M(\varepsilon, \nu) \leq N(\varepsilon, C(\varepsilon) + \nu).$$

From the above, and from the definitions of the numbers $M(\varepsilon, \nu)$ and $N(\varepsilon, \nu)$, the result is now immediate. □

We end this section with a result on (classical) topological entropy. It will be needed in the proof of Theorem 5.2. To state it, we need some terminology. Let (Y, D) be as in Lemma 3.1 and let $f : Y \longrightarrow Y$ be a continuous map. Let $K \subseteq Y$. Given $\varepsilon > 0$ and $n \in \mathbb{Z}_+$, a subset $\mathscr{F} \subset Y$ is said to *(ε, n)-span K with respect to f* if for each $x \in K$ there exists a $y \in \mathscr{F}$ so that

$$D\big(f^j(x), f^j(y)\big) \leq \varepsilon \quad \forall j = 0, 1, \dots, n-1.$$

Let $r_n(\varepsilon, K) := \inf\{\sharp\mathscr{F} : \mathscr{F} \subset Y \ (\varepsilon, n)\text{-spans } K\}$. If K is compact, then, clearly, $r_n(\varepsilon, K)$ is finite for any $\varepsilon > 0$ and $n \in \mathbb{Z}_+$. Now, define:

$$h(f, K) := \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log\big(r_n(\varepsilon, K)\big). \tag{3.8}$$

We must admit that, in the above discussion, we are omitting a considerable amount of context. The quantity $h(f, K)$ is an ingredient in the definition of Bowen's entropy $h(f)$ which — as mentioned right after Lemma 3.1 — does not require Y to be compact. We have not defined Bowen's entropy in this paper since we assume that it is widely known. However, $h(f, K)$ merits a definition as it is a more specialised quantity. Before we state the result that we need, we must mention that as Y above is compact, $h(f, K)$ does not depend on the choice of D, provided it metrises the topology on Y: see the proof of [1, Proposition 3].

Result 3.3 (Bowen, [1, Theorem 17])**.** *Let (Y_i, d_i), $i = 1, 2$, be two compact metric spaces. Let $f_i : Y_i \longrightarrow Y_i$, $i = 1, 2$, be continuous surjective maps. Let $\pi : Y_1 \longrightarrow Y_2$ be a continuous surjective map such that $\pi \circ f_1 = f_2 \circ \pi$. Then*

$$h(f_2) \leq h(f_1) \leq h(f_2) + \sup_{y \in Y_2} h(f_1, \pi^{-1}\{y\}).$$

4. The proof of Theorem 1.8

This section will chiefly be devoted to Theorem 4.2 below. Theorem 1.8 would follow as its corollary. But before we can prove Theorem 4.2, we must present an auxiliary quantity and a lemma. To do so, let X, Γ and d be as in Definition 1.3. For a given $\nu \in \mathbb{Z}_+$, **fix** a ν-tuple $\overline{\boldsymbol{\alpha}} := (\alpha_1, \dots, \alpha_\nu) \in \{1, \dots, M\}^\nu$. We say that a family $\mathscr{F}$ of ν-orbits is $(\varepsilon, \overline{\boldsymbol{\alpha}})$-separated if for all pairs of distinct elements

$$(x_0, x_1, \dots, x_\nu; \beta_1, \dots, \beta_\nu) \quad \text{and} \quad (y_0, y_1, \dots, y_\nu; \gamma_1, \dots, \gamma_\nu)$$

of $\mathscr{F}$, we have

- $(\beta_1, \dots, \beta_\nu) = \overline{\boldsymbol{\alpha}} = (\gamma_1, \dots, \gamma_\nu)$; and
- $\max_{0 \leq j \leq \nu} d(x_j, y_j) > \varepsilon$.

Lemma 4.1. *Let X and Γ be as in Definition 1.3. For $\varepsilon > 0$, $\nu \in \mathbb{Z}_+$ and $\overline{\boldsymbol{\alpha}} \in \{1, \dots, M\}^\nu$, let*

$$n(\varepsilon, \overline{\boldsymbol{\alpha}}) := \text{the cardinality of any } (\varepsilon, \overline{\boldsymbol{\alpha}})\text{-separated family of } \nu\text{-orbits having the greatest possible cardinality.}$$

Then,

$$h_{top}(\Gamma) = \sup_{\varepsilon>0} \limsup_{\nu\to\infty} \frac{1}{\nu} \log \left[\sum_{\overline{\boldsymbol{\alpha}} \in \{1,\dots,M\}^\nu} n(\varepsilon, \overline{\boldsymbol{\alpha}}) \right].$$

The proof of this lemma is extremely elementary. But since it is vital to the proof of Theorem 4.2, we provide the following

Outline of proof. Fix an $\varepsilon > 0$ and $\nu \in \mathbb{Z}_+$. For $\overline{\boldsymbol{\alpha}} \in \{1, \dots, M\}$, let $\mathscr{F}(\overline{\boldsymbol{\alpha}})$ be an $(\varepsilon, \overline{\boldsymbol{\alpha}})$-separated family such that $\sharp\mathscr{F}(\overline{\boldsymbol{\alpha}}) = n(\varepsilon, \overline{\boldsymbol{\alpha}})$. Write

$$\mathscr{F} := \bigcup_{\overline{\boldsymbol{\alpha}} \in \{1,\dots,M\}^\nu} \mathscr{F}(\overline{\boldsymbol{\alpha}}).$$

The lemma follows from the fact that $\mathscr{F}$ is an (ε, ν)-separated family and that $\sharp\mathscr{F} = N(\varepsilon, \nu)$ — where $N(\varepsilon, \nu)$ is as introduced in Proposition 3.2. Both these statements follow from the definitions and the fact that if

$$(x_0, x_1, \dots, x_\nu; \alpha_1, \dots, \alpha_\nu) \in \mathscr{F}\big((\alpha_1, \dots, \alpha_\nu)\big), \quad (y_0, y_1, \dots, y_\nu; \beta_1, \dots, \beta_\nu) \in \mathscr{F}\big((\beta_1, \dots, \beta_\nu)\big),$$

and $(\alpha_1, \dots, \alpha_\nu) \neq (\beta_1, \dots, \beta_\nu)$ then these two ν-orbits are (ε, ν)-separated in the sense of Definition 1.3. □

We can now present the central result of this section. In what follows, d_{top} will denote the topological degree, while for a surjective holomorphic map $f : X \longrightarrow X$, X a compact Kähler manifold, $d_p(f)$ will denote the p^{th} dynamical degree of $\mathsf{graph}(f)$ (see Section 2). One half of our proof of the following theorem is strongly influenced by the derivation by Misiurewicz and Przytycki [11] of a lower bound for topological entropy of a single map in the $\mathcal{C}^1$ setting. Our notation below follows the treatment of the above result by Katok–Hasselblatt in [10, Chapter 8].

Theorem 4.2. *Let (X, ω) be a compact Kähler manifold of dimension n and let S be a finitely generated semigroup consisting of surjective holomorphic self-maps of X. Let $\mathscr{G} = \{f_1, \dots, f_N\}$ be a set of generators of S. Then*

$$\begin{aligned} \log\left(\sum\nolimits_{j=1}^N d_{top}(f_j)\right) &\leq h_{top}(S, \mathscr{G}) \\ &\leq \max\left[\log(N),\, \log\left(\sum\nolimits_{j=1}^N d_{top}(f_j)\right),\, \max_{1\leq p\leq n-1} \log d_p(\Gamma_{\mathscr{G}})\right]. \end{aligned} \tag{4.1}$$

Proof. Let ω_X be the normalisation of the form ω as in Section 2. For any holomorphic correspondence Γ on X, we have (see [2, Section 3.1], for instance):

$$\lambda_n(\Gamma) = \sum\nolimits_{1\leq j\leq N} m_j \int_{\text{reg}(\Gamma_j)} \left(\pi_2|_{\Gamma_j}\right)^* \omega_X, \tag{4.2}$$

assuming the presentation (1.2) for Γ. Since, for each $j = 1, \dots, N$, (Γ_j, π_2, X) is a holomorphic branched covering, if follows from (4.2) and a change-of-variable argument that $\lambda_n(\Gamma)$ equals the topological degree of Γ: call if $d_{top}(\Gamma)$. Since the toplogical degree is multiplicative with respect to composition, it follows from (2.1) that

$$d_n(\Gamma) = \lim_{\nu\to\infty} d_{top}(\Gamma^{\circ\nu})^{1/\nu} = d_{top}(\Gamma)$$

for any holomorphic correspondence Γ on X. Applying this to the correspondence $\Gamma_{\mathscr{G}}$ we get

$$d_n(\Gamma_{\mathscr{G}}) = \sum\nolimits_{1\leq j\leq N} d_{top}(f_j). \tag{4.3}$$

A completely analogous discussion (whose details we leave to the reader) gives us $d_0(\Gamma_{\mathscr{G}}) = N$. Recalling the definition of $h_{top}(S, \mathscr{G})$, the upper bound in (4.1) follows from the last identity, (4.3), and Result 2.1.

For any holomorphic map $f : X \longrightarrow X$, let $\mathrm{Jac}(f)$ denote the real Jacobian of f determined by the volume form ω^n. Since f is holomorphic, $\mathrm{Jac}(f) \geq 0$. Fix a metric d that metrises the topology of X. Fix a number L such that

$$L > 1 \quad \text{and} \quad \sup_{x \in X} \mathrm{Jac}(f_j)(x) \leq L, \;\; j = 1, \dots, N.$$

Let us pick a number $\beta \in (0,1)$ and set $\delta(\beta) := L^{-\beta/(1-\beta)}$. Define the sets

$$\mathcal{B}(\beta, j) := \{x \in X : \mathrm{Jac}(f_j)(x) \geq \delta(\beta)\}, \quad j = 1, \dots, N,$$

and consider the open cover consisting of balls,

$$\mathscr{C}(\beta, j) := \big\{B_d(x; r_x) : x \in \mathcal{B}(\beta, j) \text{ and } f_j|_{B_d(x;r_x)} \text{ is invertible}\big\},$$

of $\mathcal{B}(\beta, j)$, $j = 1, \dots, N$. Let $\varepsilon(\beta, j) \in (0,1)$ be a Lebesgue number of $\mathscr{C}(\beta, j)$ (each $\mathcal{B}(\beta, j)$ is compact) and write $\varepsilon(\beta) := \min_{1 \leq j \leq N} \varepsilon(\beta, j)$.

Fix a $\nu \in \mathbb{Z}_+$. We simplify the symbol ${}^{\Gamma_{\mathscr{G}}}\mathscr{O}_\nu$ to ${}^{\mathscr{G}}\mathscr{O}_\nu$. For each $\overline{\boldsymbol{\alpha}} \in \{1, \dots, N\}^\nu$, define

$$A_{\beta, \overline{\boldsymbol{\alpha}}} := \big\{(x_0, x_1, \dots, x_\nu; \overline{\boldsymbol{\alpha}}) \in {}^{\mathscr{G}}\mathscr{O}_\nu \mid \sharp\{1 \leq k \leq \nu : x_{k-1} \in \mathcal{B}(\beta, \alpha_k)\} \leq \beta\nu\big\}.$$

For any l: $1 \leq l \leq \nu$, let us abbreviate

$$f_{\alpha_l} \circ \dots \circ f_{\alpha_1} =: f_{(\alpha_1, \dots, \alpha_l)}.$$

Whenever $\nu \geq 2$, the chain rule gives

$$\mathrm{Jac}\big(f_{(\alpha_1, \dots, \alpha_\nu)}\big)(x) = \left[\prod_{2 \leq k \leq \nu} \mathrm{Jac}\big(f_{\alpha_k}\big)\big(f_{(\alpha_1, \dots, \alpha_{k-1})}(x)\big)\right] \mathrm{Jac}\big(f_{\alpha_1}\big)(x).$$

Therefore, by the definitions of $A_{\beta, \overline{\boldsymbol{\alpha}}}$ and L (for any $\nu \in \mathbb{Z}_+$):

$$\begin{aligned} 0 \leq \mathrm{Jac}\big(f_{(\alpha_1, \dots, \alpha_\nu)}\big)(x_0) &< \delta(\beta)^{\nu - [\beta\nu]} L^{[\beta\nu]} \\ &\leq \delta(\beta)^{\nu(1-\beta)} L^{\beta\nu} = 1 \quad \forall (x_0, x_1, \dots, x_\nu; \overline{\boldsymbol{\alpha}}) \in A_{\beta, \overline{\boldsymbol{\alpha}}}, \end{aligned} \tag{4.4}$$

where here (and elsewhere in this proof) $[s]$ denotes the greatest integer less than or equal to s. If μ_X denotes the Borel measure, constructed in the standard manner, with the property that $\mu_X(\Omega) := \int_\Omega \omega^n$ for every coordinate patch $\Omega \subseteq X$, then:

- (*a*) (4.4) implies that $\mu_X\big(f_{(\alpha_1, \dots, \alpha_\nu)}(A_{\beta, \overline{\boldsymbol{\alpha}}})\big) < \mu_X(X)$ for each $\overline{\boldsymbol{\alpha}} = (\alpha_1, \dots, \alpha_\nu) \in \{1, \dots, N\}^\nu$.
- (*b*) Thus, if we **fix** an $\overline{\boldsymbol{\alpha}}$, then, by Sard's Theorem, there exists a point in $X \setminus f_{\overline{\boldsymbol{\alpha}}}(A_{\beta, \overline{\boldsymbol{\alpha}}})$ that is a regular value of $f_{\overline{\boldsymbol{\alpha}}}$.

Let us call this regular value x_ν.

For the $\overline{\boldsymbol{\alpha}} = (\alpha_1, \dots, \alpha_\nu)$ fixed in (*b*) above, for any α_k, $1 \leq k \leq \nu$, and any regular value y of f_{α_k}, we present a construction associated with the pair (y, k). Define

$$S(y, k) := \begin{cases} f_{\alpha_k}^{-1}\{y\}, & \text{if } f_{\alpha_k}^{-1}\{y\} \subset \mathcal{B}(\beta, \alpha_k), \\ \\ \{x^{(y)}\}, & \text{if } f_{\alpha_k}^{-1}\{y\} \not\subset \mathcal{B}(\beta, \alpha_k), \end{cases}$$

where, $x^{(y)}$ denotes some point in $f_{\alpha_k}^{-1}\{y\} \setminus \mathcal{B}(\beta, \alpha_k)$ that we pick and **fix**. We now consider the point x_ν introduced at the end of the previous paragraph. We will use it to construct a certain $(\varepsilon(\beta), \overline{\boldsymbol{\alpha}})$-separated family in ${}^{\mathscr{G}}\mathscr{O}_\nu$ using the following iterative construction. This construction is possible because, as every $f \in S$ is surjective, by the definition of a regular value we get:

$$\text{each element of } (f_{\alpha_\nu} \circ \dots \circ f_{\alpha_{k+1}})^{-1}\{x_\nu\} \text{ is a regular value of } f_{\alpha_k} \text{ for } 1 \leq k \leq \nu - 1.$$

Define (the maps appearing below were defined in Section 3):

$$\begin{aligned} {}^{1}O_{\overline{\boldsymbol{\alpha}}} &:= \{(x, x_\nu; \alpha_\nu) : x \in S(x_\nu, \nu)\}, \\ {}^{k+1}O_{\overline{\boldsymbol{\alpha}}} &:= \bigcup\nolimits_{\xi \in {}^{k}O_\nu} \big\{(x, \pi_{\text{itn}}(\xi); \alpha_{\nu-k}, \dots, \alpha_\nu) : x \in S(\mathsf{start}(\xi), \nu - k)\big\}, \ \ 1 \leq k \leq \nu - 1. \end{aligned}$$

Here, we commit a minor abuse of notation in that if, for $1 \leq k \leq \nu - 1$, ${}^{k}O_\nu \ni \xi = (x_{\nu-k}, \dots, x_\nu; \alpha_{\nu-k+1}, \dots, \alpha_\nu)$, then we interpret $(x, \pi_{\text{itn}}(\xi); \alpha_{\nu-k}, \dots, \alpha_\nu)$ to mean

$$(x, x_{\nu-k}, \dots, x_\nu; \alpha_{\nu-k}, \dots, \alpha_\nu) \ \text{ and } \mathbf{not} \ \ (x, (x_{\nu-k}, \dots, x_\nu); \alpha_{\nu-k}, \dots, \alpha_\nu).$$

With this explanation, note that each ${}^{k}O_\nu$ is a collection of k-orbits that end at the point x_ν. The iterative construction lengthens each k-orbit $\xi \in {}^{k}O_\nu$ to one or more $(k+1)$-orbits by designating new initial points for the latter orbits.

Let us write $O_{\overline{\boldsymbol{\alpha}}} := {}^{\nu}O_{\overline{\boldsymbol{\alpha}}}$. We now show that $O_{\overline{\boldsymbol{\alpha}}}$ is an $(\varepsilon(\beta), \overline{\boldsymbol{\alpha}})$-separated family of ν-orbits. To do so, consider two distinct ν-orbits

$$\mathsf{x} := (x_0, \dots, x_{\nu-1}, x_\nu; \overline{\boldsymbol{\alpha}}) \quad \text{and} \quad \mathsf{y} := (y_0, \dots, y_{\nu-1}, x_\nu; \overline{\boldsymbol{\alpha}})$$

in $O_{\overline{\boldsymbol{\alpha}}}$ (note that, by construction, the terminal points of these ν-orbits are the same). Write

$$\tau := \max\{1 \leq k \leq \nu : x_{k-1} \neq y_{k-1}\}.$$

By our iterative construction, $x_{\tau-1}, y_{\tau-1} \in f_{\alpha_\tau}^{-1}\{x_\tau\}$. In terms of the notation introduced above, this also tells us that $\sharp S(x_\tau, \tau) \geq 2$. This means that

$$x_{\tau-1}, y_{\tau-1} \in \mathcal{B}(\beta, \alpha_\tau) \quad \text{and} \quad f_{\alpha_\tau} \text{ is injective on small balls around } x_{\tau-1},\ y_{\tau-1}.$$

Clearly, $x_{\tau-1}$ and $y_{\tau-1}$ cannot belong to one single ball belonging to the open cover $\mathscr{C}(\beta, \alpha_\tau)$. Thus, by the definition of Lebesgue number, $d(x_{\tau-1}, y_{\tau-1}) > \varepsilon(\beta)$. Since $\mathsf{x} \neq \mathsf{y} \in O_{\overline{\boldsymbol{\alpha}}}$ were arbitrarily chosen, we conclude that $O_{\overline{\boldsymbol{\alpha}}}$ is an $(\varepsilon(\beta), \overline{\boldsymbol{\alpha}})$-separated family.

Write $d_j := d_{top}(f_j)$, $j = 1, \dots, N$. We may assume without loss of generality that $d_1 \leq \dots \leq d_N$. Let $\nu_j :=$ the number of times j appears in $\overline{\boldsymbol{\alpha}}$. Let us now set

$$m := [\beta\nu] + 1, \qquad J := \max\{1 \leq j \leq N : \nu_1 + \dots + \nu_j < m\}. \tag{4.5}$$

By construction, for each ν-orbit $\mathsf{x} \in O_{\overline{\boldsymbol{\alpha}}}$, $\mathsf{start}(x) \in f_{\overline{\boldsymbol{\alpha}}}^{-1}\{x_\nu\}$. Hence, by our above choice of x_ν,

$$\{\mathsf{start}(\mathsf{x}) : \mathsf{x} \in O_{\overline{\boldsymbol{\alpha}}}\} \cap A_{\beta, \overline{\boldsymbol{\alpha}}} = \varnothing. \tag{4.6}$$

Some more notation: write

$$\Sigma(\mathsf{x}) := \{1 \leq k \leq \nu : x_{k-1} \in \mathcal{B}(\beta, \alpha_k)\}, \qquad \sigma(\mathsf{x}) := \sharp\Sigma(\mathsf{x})$$

for each $\mathsf{x} = (x_0, x_1, \dots, x_\nu; \overline{\boldsymbol{\alpha}}) \in O_{\overline{\boldsymbol{\alpha}}}$. Additionally, let $k_1 < k_2 < \dots < k_{\sigma(\mathsf{x})}$ denote the ordering of the elements of $\Sigma(\mathsf{x})$. By (4.6), $\sigma(\mathsf{x}) \geq m$ for each $\mathsf{x} \in O_{\overline{\boldsymbol{\alpha}}}$. With these facts, we can estimate $\sharp O_{\overline{\boldsymbol{\alpha}}}$. To do so, pick and **fix** an $\mathsf{x} = (x_0, x_1, \dots, x_\nu; \overline{\boldsymbol{\alpha}}) \in O_{\overline{\boldsymbol{\alpha}}}$. By construction:

$$\sharp S(x_k, k) = \begin{cases} 1, & \text{if } k \notin \Sigma(\mathsf{x}), \\ d_{\alpha_k}, & \text{if } k \in \Sigma(\mathsf{x}). \end{cases}$$

This means that in $O_{\overline{\boldsymbol{\alpha}}}$:

($*$) we can find $d_{\alpha_{k_l}}$ distinct ν-orbits that traverse the points $x_{k_l}, x_{k_l+1}, \dots, x_\nu \in X$ corresponding, respectively, to iterations of orders $k_l, k_l + 1, \dots, \nu$ of $\Gamma_{\mathscr{G}}$, $l = 1, \dots \sigma(\mathsf{x})$.

This implies that $O_{\overline{\boldsymbol{\alpha}}}$ would have the smallest possible number of orbits of the kind described by ($*$) if $\nu_1 + \dots + \nu_J =: d(\overline{\boldsymbol{\alpha}})$ of the elements of $\Sigma(\mathsf{x})$ were to correspond to ν_j distinct terms in the tuple $(x_0, \dots, x_{\nu-1})$ being in $\mathcal{B}(\beta, j)$, $j = 1, \dots, J$. From this discussion and ($*$), we get the (perhaps very conservative) lower bound:

$$n(\varepsilon(\beta), \overline{\boldsymbol{\alpha}}) \geq \sharp O_{\overline{\boldsymbol{\alpha}}} \geq d_1^{\nu_1} \dots d_J^{\nu_J}\, d_{J+1}^{m - d(\overline{\boldsymbol{\alpha}})}. \tag{4.7}$$

Here, $n(\varepsilon(\beta),\overline{\boldsymbol{\alpha}})$ is as in Lemma 4.1, and the first inequality in (4.7) is owing to the fact that $O_{\overline{\boldsymbol{\alpha}}}$ is $(\varepsilon(\beta),\overline{\boldsymbol{\alpha}})$-separated.

Now, given any $(\nu_1,\dots,\nu_N) \in \mathbb{N}^N$ satisfying $\nu_1+\dots+\nu_N = [\beta\nu]+1$, we can find an $\overline{\boldsymbol{\alpha}} \in \{1,\dots,N\}^\nu$ so that the ν_j's are related to this $\overline{\boldsymbol{\alpha}}$ precisely as in the last paragraph. Thus:

$$\begin{aligned}\sum_{\overline{\boldsymbol{\alpha}}\in\{1,\dots,N\}^\nu} n(\varepsilon,\overline{\boldsymbol{\alpha}}) \;&\geq\; \sum_{\substack{\nu_1,\dots,\nu_N\in\mathbb{N}\\ \nu_1+\dots+\nu_N=m}} d_1^{\nu_1}\dots d_N^{\nu_N}\\ &=\; (d_1+\dots+d_N)^m \;\geq\; (d_1+\dots+d_N)^{\beta\nu}.\end{aligned}$$

Applying Lemma 4.1, this gives

$$h_{top}(S,\mathscr{G}) \;:=\; h_{top}(\Gamma_{\mathscr{G}}) \;\geq\; \beta\log(d_1+\dots+d_N).$$

However, as this holds for any $\beta\in(0,1)$, letting $\beta\to 1^-$, we get

$$h(S,\mathscr{G}) \;\geq\; \log(d_1+\dots d_N).$$

This establishes the lower bound in (4.1), and hence the result. $\square$

We remark here that it is, in general, not possible to get a cleaner upper bound for $h_{top}(S,\mathscr{G})$ than (4.1). For instance, there isn't, in general, a way to determine which of the numbers $\{d_1(f),\dots,d_n(f)\}$ is the largest even for $f: X\longrightarrow X$ surjective and holomorphic (let alone for a general correspondence Γ). We shall not discuss here what **is** known in general about the function $\{0,1,\dots,n\}\ni p\mapsto d_p(f)$. However, for X as above, $\lambda_p(f)$, for $f: X\longrightarrow X$ holomorphic and $p=0,1,\dots,n$, can be determined cohomologically. This can lead to cleaner expressions whenever $H^{p,p}(X;\mathbb{R})$ are one-dimensional for each $p=1,\dots,n$. This, essentially, is what underlies

The proof of Theorem 1.8. Russakovskii–Shiffman have shown [12, Section 4] that for any non-constant holomorphic map $f:\mathbb{P}^n\longrightarrow\mathbb{P}^n$

$$\lambda_p(f) \;=\; d_p(f) \quad\text{and}\quad d_p(f)\;=\;d_1(f)^p \quad \text{for } p=1,\dots,n. \tag{4.8}$$

As argued in the proof of Theorem 4.2, $d_n(f)=d_{top}(f)$. Thus, from the above facts, we get

$$\lambda_p(f) \;=\; d_{top}(f)^{p/n}, \quad \text{for } p=1,\dots,n. \tag{4.9}$$

Fix a set of generators $\{f_1,\dots,f_N\}$ of S. Clearly, by definition, $\lambda_p(\Gamma_{\mathscr{G}}^{\circ\nu})$ is the sum of the p^{th} intermediate degrees of the maps, counted according to multiplicity, whose graphs constitute $\Gamma_{\mathscr{G}}^{\circ\nu}$. From (4.9), we see that for $\mathbb{P}^n$, λ_p, $p=1,\dots,n$, is multiplicative with respect to composition of non-constant holomorphic self-maps. Thus, by (4.8), we get

$$\lambda_p(\Gamma_{\mathscr{G}}^{\circ\nu}) \;=\; \big(d_1(f_1)^p+\dots+d_1(f_N)^p\big)^\nu \quad \text{for } p=1,\dots,n.$$

Hence, $d_p(\Gamma_{\mathscr{G}}) = \big(d_1(f_1)^p+\dots+d_1(f_N)^p\big)$, $p=1,\dots,n$. As for $p=0$: $d_0(\Gamma_{\mathscr{G}})=N$. Given these facts, the conclusion of Theorem 1.8 follows from Theorem 4.2. $\square$

Corollary 1.9 now follows immediately.

5. Concerning Friedland's entropy

This section is dedicated to the result on h_F mentioned in Section 1 — i.e., Theorem 5.2. First, however, we need some notation and a lemma. Let S be a finitely generated rational semigroup on $\mathbb{P}^1$. If we fix a finite set of generators $\mathscr{G}$, then the space Γ^∞ (introduced in Definition 1.2) corresponding to $\Gamma_{\mathscr{G}}$ will be denoted by $\Gamma^\infty_{\mathscr{G}}$. Also, we abbreviate ${}^{\Gamma_{\mathscr{G}}}\mathbb{P}^1$ to ${}^{\mathscr{G}}\mathbb{P}^1$. Given any holomorphic correspondence Γ from X_1 to X_2, where X_i, $i=1,2$, are as in Definition 1.1, we define

$$F_\Gamma(x) \;:=\; \pi_2\Big(\big(\,\pi_1|_{|\Gamma|}\big)^{-1}\{x\}\Big) \quad \forall x\in X_1,$$

and write $F^{\nu}_{\Gamma}(x) := F_{\Gamma^{\circ\nu}}(x)$. Coming back to the correspondence $\Gamma_{\mathscr{G}}$: we abbreviate $F^{\nu}_{\Gamma_{\mathscr{G}}}(x)$ to $F^{\nu}_{\mathscr{G}}(x)$.

Lemma 5.1. *Let S be a finitely generated rational semigroup on $\mathbb{P}^1$. Let $\mathscr{G} = \{f_1, \dots, f_N\}$ be a set of generators of S. Write*

$$\mathfrak{S}_{\mathscr{G}} := \bigcup_{j=1}^{N} \bigcup_{i \neq j} \{x \in \mathbb{P}^1 : f_i(x) = f_j(x)\}.$$

Consider a point $\mathscr{O} = (x_0, x_1, x_2, \dots) \in \Gamma^{\infty}_{\mathscr{G}}$. If the pre-image of $\mathscr{O}$ under the map $\pi_{\text{itn}} : {}^{\mathscr{G}}\mathbb{P}^1 \longrightarrow \Gamma^{\infty}_{\mathscr{G}}$ is infinite, then there exist an $n^{\bullet} \in \mathbb{N}$ and a point

$$x^{\bullet} \in \mathfrak{S}_{\mathscr{G}} \bigcap \Big(\limsup_{\nu \to \infty} F^{\nu}_{\mathscr{G}}(x^{\bullet}) \Big)$$

such that $x^{\bullet} = x_{n^{\bullet}}$.

Proof. First consider any $\mathscr{O} = (x_0, x_1, x_2, \dots) \in \Gamma^{\infty}_{\mathscr{G}}$. For each $\nu \in \mathbb{Z}_+$, there are only finitely many j such that $f_j(x_{\nu-1}) = x_{\nu}$. Thus, it is easy to see that $\pi^{-1}_{\text{itn}}\{\mathscr{O}\}$ is infinite if and only if

($**$) there exists a sequence of positive integers $\nu_1 < \nu_2 < \nu_3 < \cdots$ such that for each $k \in \mathbb{Z}_+$, $f_j(x_{(\nu_k - 1)}) = x_{\nu_k}$ for **more than** one $j \in \{1, \dots, N\}$.

Now, assume that $\pi^{-1}_{\text{itn}}\{\mathscr{O}\}$ is infinite. Then, by ($**$) there exists a sequence of positive integers $\nu_1 < \nu_2 < \nu_3 < \cdots$ such that

$$x_{(\nu_k - 1)} \in \mathfrak{S}_{\mathscr{G}} \quad \forall k \in \mathbb{Z}_+. \tag{5.1}$$

Since $f_1, \dots, f_N$ are distinct and $\mathbb{P}^1$ is one-dimensional, $\mathfrak{S}_{\mathscr{G}}$ is finite. Thus, by (5.1), we conclude that there exists an increasing subsequence $\{\nu_{k_\ell}\} \subset \{\nu_k\}$ and a point $x^{\bullet} \in \mathfrak{S}_{\mathscr{G}}$ such that

$$x_{(\nu_{k_\ell} - 1)} = x^{\bullet} \in \mathfrak{S}_{\mathscr{G}} \quad \forall \ell \in \mathbb{Z}_+.$$

If we write $n_\ell := \nu_{k_\ell} - \nu_{k_1}$, then the above equation implies that $x^{\bullet} \in F^{n_\ell}_{\mathscr{G}}(x^{\bullet})$ for every $\ell \in \mathbb{Z}_+ \setminus \{1\}$. Therefore, we conclude that

$$x^{\bullet} \in \bigcap_{k \in \mathbb{N}} \bigcup_{\nu \geq k} F^{\nu}_{\mathscr{G}}(x^{\bullet}) =: \limsup_{\nu \to \infty} F^{\nu}_{\mathscr{G}}(x^{\bullet}).$$

Taking $n^{\bullet} = (\nu_{k_1} - 1)$, the desired conclusion is obtained. $\square$

Before we present Theorem 5.2, we elaborate upon the remark made towards the end of Section 1. For the purposes of this discussion, let X be any compact metric space and let S be the semigroup generated by the maps $f_j : X \longrightarrow X$, $j = 1, \dots, N$, that are *continuous*, and take $\Gamma = \cup_{1 \leq j \leq N} \mathsf{graph}(f_j)$ in Definition 1.2. For each $A \subsetneq \{1, \dots, N\}$, $A \neq \varnothing$, consider

$$Y^{(A)} := \{(x_0, x_1, x_2, \dots) \in \Gamma^{\infty} : x_{n+1} = f_j(x_n) \text{ for some } j \in A,\ n = 0, 1, 2, \dots\}.$$

$Y^{(A)}$ is a closed subspace of Γ^{∞} that is invariant under σ, where σ is as in Definition 1.2. Recalling the definition of $h_F(S, \mathscr{G})$, the basic properties of Bowen's entropy, and as $Y^{(A)}$ is σ-invariant, we get (as before, $\mathscr{G} := \{f_1, \dots, f_N\}$)

$$h_F(S, \mathscr{G}) \geq h\big(\sigma|_{Y^{(A)}} \big) = h_F(\langle f_j : j \in A \rangle, \{f_j : j \in A\}).$$

When $A = \{j\}$, write $Y^{(A)} =: Y^{(j)}$. Observe: f_j is conjugate to $\sigma|_{Y^{(j)}}$ via the map $x \longmapsto (x, f_j(x), f_j \circ f_j(x), \dots) \in Y^{(j)}$. From this and our preceding argument, we get

$$h_F(S, \mathscr{G}) \geq h\big(\sigma|_{Y^{(j)}} \big) = h(f_j) \quad \forall j = 1, \dots, N.$$

Hence, $h_F(S, \mathscr{G}) \geq \max_{1 \leq j \leq N} h(f_j)$. In particular, this estimate holds true for all the semigroups discussed in Sections 1, 4 and the present section.

However, one might intuit from the previous lemma that the latter lower bounds are in general inefficient (as least when S is a finitely generated rational semigroup on $\mathbb{P}^1$). That intuition motivates the principal result of this section. We follow below the notation established for Proposition 3.2 — for instance, $\mathscr{S}$ is the shift map introduced by that proposition.

Theorem 5.2. *Let S be a finitely generated rational semigroup on $\mathbb{P}^1$. Let $\mathscr{G} = \{f_1, \dots, f_N\}$ be a set of generators of S. Define*

$$\mathcal{E}(\mathscr{G}) \,:=\, \Big\{(x_0, x_1, x_2, \dots) \in \Gamma^\infty_{\mathscr{G}} : x_0 \in \mathfrak{S}_{\mathscr{G}} \text{ and } x_0 \in \limsup_{\nu\to\infty} F^\nu_{\mathscr{G}}(x_0)\Big\}.$$

Then, Friedland's entropy satisfies

$$\log\Big(\sum\nolimits_{1\leq j\leq N} \deg(f_j)\Big) - \sup_{\mathscr{O}\in\mathcal{E}(\mathscr{G})} h\big(\mathscr{S}, \pi^{-1}_{\rm itn}\{\mathscr{O}\}\big) \,\leq\, h_F(S, \mathscr{G}) \\ \leq\, \log\Big(\sum\nolimits_{1\leq j\leq N} \deg(f_j)\Big).$$

Proof. Consider the compact metric spaces ${}^{\mathscr{G}}\mathbb{P}^1$ and $\Gamma^\infty_{\mathscr{G}}$. Denote by $\overline{\sigma} : \Gamma^\infty_{\mathscr{G}} \longrightarrow \Gamma^\infty_{\mathscr{G}}$ the shift

$$\Gamma^\infty_{\mathscr{G}} \ni (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots).$$

Recall that $\mathscr{S} : {}^{\mathscr{G}}X \longrightarrow {}^{\mathscr{G}}X$ is as described in Proposition 3.2. We have:

- $\pi_{\rm itn}$ is a continuous surjective map; and
- $\pi_{\rm itn} \circ \mathscr{S} = \overline{\sigma} \circ \pi_{\rm itn}$.

In other words, $\overline{\sigma}$ is a factor of $\mathscr{S}$. Thus, Proposition 3.2 and Corollary 1.9 together imply:

$$h_F(S, \mathscr{G}) \,:=\, h(\overline{\sigma}) \,\leq\, h(\mathscr{S}) \,=\, h_{top}(S, \mathscr{G}) \,=\, \log\Big(\sum\nolimits_{1\leq j\leq N} \deg(f_j)\Big). \tag{5.2}$$

We now derive the lower bound for h_F. For the moment, **fix** an $\mathscr{O} \in \Gamma^\infty_{\mathscr{G}}$. We have two cases.

Case 1. *$\pi^{-1}_{\rm itn}\{\mathscr{O}\}$ is a finite set.*
For any $\nu \in \mathbb{Z}_+$ and $\varepsilon > 0$, $\mathscr{O}$ (ε, ν)-spans itself. Thus, the finiteness of $\pi^{-1}_{\rm itn}\{\mathscr{O}\}$ implies that $h(\mathscr{S},\, \pi^{-1}_{\rm itn}\{\mathscr{O}\}) = 0$.

Case 2. *$\pi^{-1}_{\rm itn}\{\mathscr{O}\}$ is an infinite set.*
Write $\mathscr{O} = (x_0, x_1, x_2, \dots)$. In this case, Lemma 5.1 enables us to define

$$k(\mathscr{O}) \,:=\, \min\Big\{n \in \mathbb{N} : x_n \in \mathfrak{S}_{\mathscr{G}} \cap \big(\limsup_{\nu\to\infty} F^\nu_{\mathscr{G}}(x_n)\big)\Big\}.$$

If $k(\mathscr{O}) \geq 1$, then by definition, there is a fixed tuple $(\alpha_1, \dots, \alpha_{k(\mathscr{O})})$ such that every element of $\pi^{-1}_{\rm itn}\{\mathscr{O}\}$ has the form $(x_0, \dots, x_{k(\mathscr{O})}, \dots; \alpha_1, \dots, \alpha_{k(\mathscr{O})}, \dots)$. Thus, following the notation of the discussion that precedes Result 3.3, for each $\varepsilon > 0$ we have:

$$r_\nu\big(\varepsilon, \pi^{-1}_{\rm itn}\{\mathscr{O}\}\big) = r_\nu\big(\varepsilon, \pi^{-1}_{\rm itn}\{(x_{k(\mathscr{O})},\, x_{k(\mathscr{O})+1},\, x_{k(\mathscr{O})+2}\,, \dots)\}\big) \quad \forall \nu \text{ suffiiciently large.}$$

We therefore conclude (irrespective of whether $k(\mathscr{O}) = 0$ or $k(\mathscr{O}) \geq 1$) that

$$h\big(\mathscr{S}, \pi^{-1}_{\rm itn}\{\mathscr{O}\}\big) \,=\, h\big(\mathscr{S}, \pi^{-1}_{\rm itn}\{(x_{k(\mathscr{O})},\, x_{k(\mathscr{O})+1},\, x_{k(\mathscr{O})+2}\,, \dots)\}\big), \text{ and} \\ x_{k(\mathscr{O})} \,\in\, \mathfrak{S}_{\mathscr{G}} \cap \big(\limsup_{\nu\to\infty} F^\nu_{\mathscr{G}}(x_{k(\mathscr{O})})\big). \tag{5.3}$$

From the discussion of each of the above cases, and by (5.3), we get

$$\sup_{\mathscr{O}\in\Gamma^\infty_{\mathscr{G}}} h\big(\mathscr{S}, \pi^{-1}_{\rm itn}\{\mathscr{O}\}\big) \,=\, \sup_{\mathscr{O}\in\mathcal{E}(\mathscr{G})} h\big(\mathscr{S}, \pi^{-1}_{\rm itn}\{\mathscr{O}\}\big).$$

From this, Result 3.3 and Proposition 3.2, we have

$$h_{top}(S, \mathscr{G}) \,=\, h(\mathscr{S}) \,\leq\, h(\overline{\sigma}) + \sup\nolimits_{\mathscr{O}\in\mathcal{E}(\mathscr{G})} h\big(\mathscr{S}, \pi^{-1}_{\rm itn}\{\mathscr{O}\}\big) \\ = h_F(S, \mathscr{G}) + \sup\nolimits_{\mathscr{O}\in\mathcal{E}(\mathscr{G})} h\big(\mathscr{S}, \pi^{-1}_{\rm itn}\{\mathscr{O}\}\big). \tag{5.4}$$

From (5.2) and (5.4), and given the conclusion of Corollary 1.9, the theorem follows. □

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