SURFACES WITH DEGENERATE CR SINGULARITIES THAT ARE LOCALLY POLYNOMIALLY CONVEX

GAUTAM BHARALI

1. Introduction and Statement of Results

A compact subset $K \subset \mathbb{C}^n$ is said to be **polynomially convex** if for every point $\zeta \notin K$, there exists a holomorphic polynomial P such that $P(\zeta) = 1$ and $\sup_K |P| < 1$. K is said to be **locally polynomially convex** at a point $p \in K$ if there exists a closed ball $\mathbb{B}(p)$ centered at p such that $K \cap \mathbb{B}(p)$ is polynomially convex. In general, it is difficult to determine whether a given compact $K \subset \mathbb{C}^n$ is polynomially convex when $n > 1$. In contrast, there is a considerable body of work concerning the (local) polynomial convexity of smooth surfaces in $\mathbb{C}^n$. Let $\mathfrak{S}$ be a smooth real surface $\mathfrak{S}$ in $\mathbb{C}^n$, $n > 1$. A point $p \in \mathfrak{S}$ is said to be **totally real** if the tangent plane $T_p(\mathfrak{S})$ at p is not a complex line. A point on $\mathfrak{S}$ that is not totally real will be called a **CR singularity**. At a totally real point $p \in \mathfrak{S}$, the surface $\mathfrak{S}$ is locally polynomially convex. This is not always the case if $p \in \mathfrak{S}$ is an isolated CR singularity. When, at a CR singularity $p \in \mathfrak{S}$, the order of contact of $T_p(\mathfrak{S})$ with $\mathfrak{S}$ equals 2, the situation is well understood. Suppose $\mathfrak{S} \subset \mathbb{C}^2$; then there exist local holomorphic coordinates (z, w) with respect to which $p = (0, 0)$, and such that $\mathfrak{S}$ is locally given by an equation of the form $w = |z|^2 + \gamma(z^2 + \overline{z}^2) + F(z)$, where $\gamma > 0$ and $F(z) = O(|z|^3)$. In Bishop's terminology, the CR singularity $p = (0, 0)$ is said to be elliptic if $0 < \gamma < 1/2$, parabolic if $\gamma = 1/2$ and hyperbolic is $\gamma > 1/2$. Bishop showed [1] that if p is elliptic, then given $\varepsilon_0 > 0$ there is a one-parameter family of analytic discs whose boundaries are contained in $\mathfrak{S} \cap \mathbb{B}(p; \varepsilon_0)$, whence $\mathfrak{S}$ is *not* polynomially convex. Much later, Forstnerič & Stout [3] showed that if $p \in \mathfrak{S}$ is an isolated, hyperbolic CR singularity, then $\mathfrak{S}$ is locally polynomially convex at p.

Very litte is known if the order of contact of $T_p(\mathfrak{S})$ with $\mathfrak{S}$ at a CR singularity p is *greater* than 2. We will call such a CR singularity a **degenerate CR singularity**. The aim of this paper is to study when $\mathfrak{S}$ is locally polynomially convex at an isolated, degenerate CR singularity. Knowing so may be useful in function theory : for instance, if a surface $\mathfrak{S}$ had only isolated CR singularities, and one knew that $\mathfrak{S}$ was locally polynomially convex at each singularity, then $\mathfrak{S}$ would have a Stein neighbourhood basis. Local polynomial convexity at a degenerate CR singularity may be inferred in some cases when $\mathfrak{S}$ is the graph of a function F^* and $F^* : \mathbb{C} \to \mathbb{C}$ is a globally-defined, finitely-sheeted branched covering; refer to [7] for a precise statement. It would, however, be useful to deduce local polynomial convexity using merely local information. To this end, we provide certain sufficient conditions for a smooth surface $\mathfrak{S}$ to be locally polynomially convex at an

---

2000 *Mathematics Subject Classification.* Primary 32E20.

Key words and phrases. Complex tangency, CR singularity, polynomially convex.

Appeared in ***Michigan Math. J.* 53** (2005), 429-445.

isolated, degenerate CR singularity. Given a compact subset $K \subset \mathbb{C}^n$, let $\mathcal{P}(K)$ denote the function space of uniform limits of the holomorphic polynomials on K. Questions about the polynomial convexity of K are closely related to whether $\mathcal{P}(K) = \mathcal{C}(K)$. In particular, for any compact $K \subset \mathbb{C}^n$, $\mathcal{P}(K) = \mathcal{C}(K)$ implies that K is polynomially convex (we will justify this assertion in §3). Here, and in what follows, $\mathcal{C}(K)$ will denote the class of *complex*-valued continuous functions on K. Our results provide sufficient conditions showing that, given a surface $\mathfrak{S}$ and an isolated CR singularity $p \in \mathfrak{S}$, not only is there a small compact $\mathfrak{S}$-neighbourhood of p that is polynomially convex, but that all continuous functions on this portion of $\mathfrak{S}$ can be approximated uniformly by holomorphic polynomials.

We now state our first result :

Theorem 1.1. *Let $\mathfrak{S}$ be a smooth surface in $\mathbb{C}^2$ given by*

$$w \ = \ \sum_{\alpha+\beta=k} C_{\alpha,\beta} z^{\alpha} \overline{z}^{\beta} + F(z),$$

where (z, w) are holomorphic coordinates on $\mathbb{C}^2$, F is a smooth function satisfying $F(z) = o(|z|^k)$ as $z \to 0$, and $k > 2$. Assume that $\mathfrak{S}$ has an isolated CR singularity at the origin. Let us write

$$\sum_{\alpha+\beta=k} C_{\alpha,\beta} z^{\alpha} \overline{z}^{\beta} \ = \ C_{k,0} z^k + C_{0,k} \overline{z}^k + \Sigma(z), \qquad C_{0,k} \neq 0.$$

If $|\Sigma(z)| \leq \kappa |z|^k$ for some κ satisfying

$$0 \ \leq \ \kappa \ < |C_{0,k}| \min\left\{ \frac{\pi}{2k}, \frac{1}{2} \right\}, \tag{1.1}$$

then there exists a constant $\varepsilon_0 > 0$ such that $\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ is polynomially convex. Furthermore, $\mathcal{P}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$.

One may ask whether there is a purely geometric condition such as hyperbolicity – as opposed to the analytical condition above – according to which a surface $\mathfrak{S}$ is locally polynomially convex at a degenerate CR singularity $p \in \mathfrak{S}$. The Maslov index (the reader is referred to [2] for a definition) is an invariant associated with an isolated CR singularity. Elliptic CR singularities have Maslov index 1 while hyperbolic points have Maslov index -1. In view of the definition of the Maslov index, it is reasonable to ask if a surface is locally polynomially convex at an isolated, degenerate CR singularity with *negative* Maslov index. However, this is not true. An example of Wiegerinck [9, Example 4.3] shows that a surface can have a non-trivial polynomial hull near an isolated, degenerate CR singularity with negative Maslov index. It thus seems that additional conditions are necessary.

Now, the analytical condition in Theorem 1.1 essentially says that if a surface $\mathfrak{S}$ is, around a CR singularity taken to be the origin, presented as a graph of a function F^* with leading order $k > 2$, then $\mathfrak{S}$ is locally polynomially convex if the Taylor coefficients of all the leading terms of F^* *other than* the $\overline{z}^k$ term are in some sense small in comparison to that of the $\overline{z}^k$ term. However, by adapting the technique of Forstnerič & Stout to the case of degenerate CR singularities, one can also demonstrate local polynomial convexity in cases where some of the leading Taylor coefficients of the graphing function are not small in comparison to that of the $\overline{z}^k$ term. This is the situation addressed by the following theorem :

Theorem 1.2. *Let $\mathfrak{S}$ be a smooth surface in $\mathbb{C}^2$ given by*

$$w = \sum_{\alpha+\beta=2k} C_{\alpha,\beta} z^\alpha \overline{z}^\beta + \widetilde{F}(z),$$

where (z, w) are holomorphic coordinates on $\mathbb{C}^2$, $\widetilde{F}$ is a smooth function satisfying $\widetilde{F}(z) = o(|z|^{2k})$ as $z \to 0$, and $k > 1$. Assume that $\mathfrak{S}$ has an isolated CR singularity at the origin. Let us write

$$\sum_{\alpha+\beta=2k} C_{\alpha,\beta} z^\alpha \overline{z}^\beta = C_{2k,0} z^{2k} + C_{k,k}|z|^{2k} + C_{0,2k}\overline{z}^{2k} + \widetilde{\Sigma}(z),$$

$$\gamma = \frac{|C_{0,2k}|}{|C_{k,k}|} .$$

If $\gamma > 1/2$ and $|\widetilde{\Sigma}(z)| \leq \kappa(2\gamma - 1)|z|^{2k}$ for some κ satisfying

$$0 \leq \kappa < \frac{|C_{k,k}|}{2} \min\left\{\frac{\pi}{2k}, \frac{2\gamma - 1}{2\gamma(3\gamma + 2)}\right\}, \tag{1.2}$$

then there exists a constant $\varepsilon_0 > 0$ such that $\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ is polynomially convex. Furthermore, $\mathcal{P}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$.

We do not claim that the condition (1.2) is the best possible condition that guarantees local polynomial convexity. On the other hand, if $\widetilde{\Sigma}(z) = 0$, then an obvious modification of the arguments in [1] shows that $\mathfrak{S}$ would *not* be polynomially convex if $\gamma < 1/2$. The case $\gamma = 1/2$ leads to varying phenomena as in the case when $0 \in \mathbb{C}^2$ is a non-degenerate CR singularity (the interested reader is refered to [5] on the last subject).

2. Some notation and remarks

The primary purpose of this section is to state Kallin's lemma [6], which is instrumental in demonstrating (local) polynomial convexity of various configurations in $\mathbb{C}^n$, $n > 1$, and to remark upon its connection with our results. We state a form of Kallin's lemma that we shall use in Sections 3 & 4; the reader is referred to [6] for the original result.

Lemma 2.1 (Kallin)**.** *Suppose X_1 and X_2 are compact subsets of $\mathbb{C}^n$ such that $\mathcal{P}(X_j) = \mathcal{C}(X_j)$, $j = 1, 2$. Let $\phi : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic polynomial such that $\phi(X_j) \subset W_j$, $j = 1, 2$, where W_1 and W_2 are polynomially convex compact sets in $\mathbb{C}$ and $W_1 \cap W_2 = \{0\}$. Assume that $\phi^{-1}\{0\} \cap (X_1 \cup X_2) = X_1 \cap X_2$. Then $\mathcal{P}(X_1 \cup X_2) = \mathcal{C}(X_1 \cup X_2)$.*

The above version of Kallin's lemma is implicit in the proof of Theorem IV in [3]. But, for the reader's convenience, we provide the following :

Sketch of the proof of Lemma 2.1. The conditions on W_j, $j = 1,2$, imply that W_1 and W_2 are peak sets for $\mathcal{P}(W_1 \cup W_2)$. As $\mathcal{P}(X_1 \cup X_2)$ is a closed subspace of $\mathcal{C}(X_1 \cup X_2)$, there is a regular Borel measure μ on $X_1 \cup X_2$ that annihilates $\mathcal{P}(X_1 \cup X_2)$. Define $\mu_j := \mu|_{X_j}$, $j = 1,2$. Let $f \in \mathcal{P}(W_1 \cup W_2)$ peak on W_1. Then, for every holomorphic polynomial P on $\mathbb{C}^n$, we have

$$0 \;=\; \lim_{\nu\to\infty} \int_{X_1 \cup X_2} (f \circ \phi)^\nu P \, d\mu \;=\; \int_{X_1} P \, d\mu,$$

whence μ_1 annihilates $\mathcal{P}(X_1) = \mathcal{C}(X_1)$, which implies that $\mu_1 = 0$. Similarly, $\mu_2 = 0$. We have shown that $\mu = 0$. Therefore $\mathcal{P}(X_1 \cup X_2) = \mathcal{C}(X_1 \cup X_2)$. $\square$

In our proofs, we will extend a technique presented in [3, Theorem IV]. In the proofs of both the results, we will find an appropriate proper polynomial mapping of $\mathbb{C}^2$ onto $\mathbb{C}^2$ such that the pre-image of an appropriately small compact $\mathfrak{S}$-neighbourhood of the origin under this proper mapping is a finite union of compact subsets $X_1, \dots, X_N$ which satisfy $\mathcal{P}(X_j) = \mathcal{C}(X_j)$, $j = 1, 2, \dots, N$. $N = k$ in Theorem 1.1, and $N = 2$ in Theorem 1.2. We will then show that the sets $X_1, \dots, X_N$ are mapped by a polynomial into distinct sectors in $\mathbb{C}$, which intersect only at the origin. It is at this stage that one needs Lemma 2.1, and one infers that $\mathcal{P}(X_1 \cup \dots \cup X_N) = \mathcal{C}(X_1 \cup \dots \cup X_N)$. The desired conclusions follow from the last statement by appealing to the theory analytic covers. These proofs are presented in the next section. The proof of Theorem 1.2 incorporates the use of certain technical lemmas. The proofs of those lemmas have been deferred to §4.

Before presenting the proofs of our results, we define a couple of concepts that will be used in §3. Firstly, if K is a compact subset of $\mathbb{C}^n$, the **polynomially convex hull of $\boldsymbol{K}$**, written $\widehat{K}$, is defined by

$$\widehat{K} \;:=\; \{\zeta \in \mathbb{C}^n \mid |P(\zeta)| < \sup_K |P|, \text{ for every holomorphic polynomial } P\}.$$

Given a uniform algebra $\mathcal{A}$, the **maximal ideal space of $\boldsymbol{\mathcal{A}}$** is the space of all algebra-homomorphisms of $\mathcal{A}$ to $\mathbb{C}$, viewed as a subspace of the dual space $\mathcal{A}^*$ with the weak* topology (it is a standard fact that every complex homomorphism of $\mathcal{A}$ is in fact continuous). Recall that for a compact subset K, the maximal ideal space of $\mathcal{C}(K)$ is homeomorphically identified with K. We will need this fact in §3.

3. Proof of the Main Results

The proof of Theorem 1.1. We begin by noting that we may assume, without loss of generality, that $C_{k,0} = 0$. This is because when this is not the case, we can simply choose new holomorphic coordinates (z^*, w^*) given by

$$z^* \;:=\; z, \qquad w^* \;:=\; w - C_{k,0} z^k,$$

whereby the coefficient of the $(z^*)^k$ term of the function defining $\mathfrak{S}$ vanishes. For this reason, we shall assume, in the following argument, that $C_{k,0} = 0$. Let $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\Psi(z,w) := (z, w^k)$. This is a proper map of $\mathbb{C}^2$ onto itself having multiplicity k. Notice that there exists a $\delta > 0$ such that $\Psi^{-1}(\{(z,w) : |z| \leq \delta\} \cap \mathfrak{S}) = \cup_{j=1}^{k} \mathsf{S}_j(\delta)$, where $\mathsf{S}_1(\delta), \dots, \mathsf{S}_k(\delta)$, are compact sets given by the equations

$$\mathsf{S}_j(\delta) \ : \quad w \ = \ e^{2\pi i(j-1)/k} \ c^* \overline{z} \ \{1 + H(z)\}, \quad |z| \leq \delta,$$

where $c^* := |C_{0,k}| e^{i\mathrm{Arg}(C_{0,k})/k}$, and H is a continuous function satisfying a useful estimate. To justify this statement, we introduce the function F^* and view the surface $\mathfrak{S}$ as the graph of F^*. F^* may be written as

$$F^*(z) \ = \ C_{0,k}\overline{z}^k \ \left\{ 1 + \frac{\Sigma(z)}{C_{0,k}\overline{z}^k} + \frac{F(z)}{C_{0,k}\overline{z}^k} \right\}.$$

Observe that

a) Owing to the estimate for $\Sigma(z)$ and the condition (1.1)

$$\left| \ \frac{\Sigma(z)}{C_{0,k}\overline{z}^k} \ \right| \ \leq \ \frac{\kappa}{|C_{0,k}|} \ < \ \frac{1}{2}; \text{ and}$$

b) $\lim_{z\to 0} F(z)/C_{0,k}\overline{z}^k = 0$.

For the above reasons, we can find a $\delta > 0$ so small that

$$\left| \ \frac{\Sigma(z)}{C_{0,k}\overline{z}^k} + \frac{F(z)}{C_{0,k}\overline{z}^k} \ \right| \ \leq \ \frac{1}{2} \quad \forall |z| \leq \delta.$$

Given this fact, $F^*(z)$ has k distinct k^{th}-roots $f_1^*(z), \dots, f_k^*(z)$ when $0 < |z| \leq \delta$, each $\mathsf{S}_j(\delta)$ is the graph of f_j^*, $j = 1, \dots, k$, and

$$f_j^*(z) \ = \ e^{2\pi i(j-1)/k} \ c^* \overline{z} \ \left[\ 1 + \sum_{m=1}^{\infty} \alpha_m \left\{ \frac{\Sigma(z)}{C_{0,k}\overline{z}^k} \right\}^m + o(1) \ \right], \tag{3.1}$$

where the α_m's are the coefficients occuring in the Taylor expansion of $(1+x)^{1/k}$ around $x = 0$, i.e.

$$(1+x)^{1/k} \ = \ 1 + \sum_{m=1}^{\infty} \alpha_m x^m, \quad |x| < 1.$$

The infinite series in equation (3.1) represents a function h that is homogeneous of degree 0, and we write $H(z) := h(z) + R(z)$, where $R(z) = o(1)$ as $z \to 0$. Notice that

$$\sup_{|z|=1} |h(z)| \ \leq \ \sum_{m=1}^{\infty} |\alpha_m| \left\{ \min\left(\frac{1}{2}, \frac{\pi}{2k}\right) \right\}^m \ = \ 1 - \left\{ 1 - \min\left(\frac{1}{2}, \frac{\pi}{2k}\right) \right\}^{1/k} \ < \ \min\left(\frac{1}{2}, \frac{\pi}{2k}\right). \tag{3.2}$$

The last inequality above follows from the fact that $k > 2$ and that the term in the braces is smaller than 1. In view of (3.1) and (3.2) and the second part of condition (1.1), it is possible to find a small constant $\varepsilon_1 > 0$ such that

$$\begin{aligned} |\overline{z}H(z) - \overline{\zeta}H(\zeta)| \ &\leq \ \sup_{|\xi|=1} |h(\xi)| \ |z - \zeta| + |z| \ |h(z) - h(\zeta)| + |\overline{z}R(z) - \overline{\zeta}R(\zeta)| \\ &< \ |z - \zeta| \quad \forall z \neq \zeta \ : \ |z|, |\zeta| \leq \delta, \text{ and } \forall \delta \in (0, \varepsilon_1]. \end{aligned} \tag{3.3}$$

The estimate (3.3) allows us to use a result of Wermer [8, Theorem 1] to conclude that for each $\mathsf{S}_j(\delta)$, $\mathcal{P}(\mathsf{S}_j(\delta)) = \mathcal{C}(\mathsf{S}_j(\delta))$, $j = 1, \dots, k$, assuming, of course, that $0 < \delta \leq \varepsilon_1$.

Consider the polynomial $p(z,w)=zw/c^*$. For any $(z,w)\in\mathsf{S}_1(\delta)$,

$$\begin{aligned}\mathrm{Re}\{p(z,w)\} &= |z|^2+\mathrm{Re}\{\,|z|^2(h(z)+R(z))\} \;\geq\; |z|^2-|z|^2|h(z)|-|z|^2|R(z)|,\\ |\mathrm{Im}\{p(z,w)\}| &\leq |z|^2\{\,|h(z)|+|R(z)|\,\}.\end{aligned}$$

In view of the estimate (3.2) and the fact that $R(z)=o(1)$ as $z\to 0$, we can find a number M satisfying $1/2<M<1-\kappa/|C_{0,k}|$ and a small constant $\varepsilon_2>0$ such that

$$\mathrm{Re}\{p(z,w)\} \;\geq\; M|z|^2, \qquad |\mathrm{Im}\{p(z,w)\}| \;<\; \frac{\pi}{2k}|z|^2 \quad \forall |z|\leq\delta,$$

$$p(\mathsf{S}_1(\delta)) \;\subsetneq\; \{x+iy\in\mathbb{C}:|y|\leq(\pi/2kM)x\}, \quad \text{where } \delta\in(0,\varepsilon_2]. \tag{3.4}$$

(3.4) says that $p(\mathsf{S}_1(\delta))$ is a proper subset of the sector W_1 that is centered on the positive x-axis, and has an aperture of (π/kM). Note, therefore that $p(\mathsf{S}_j(\delta))$ is a proper subset of the sector W_j, which is simply a copy of W_1 rotated by $(2\pi(j-1)/k)$, $j=1,\dots,k$.

We have shown so far that :

- For each $\mathsf{S}_j(\delta)$, $\mathcal{P}(\mathsf{S}_j(\delta))=\mathcal{C}(\mathsf{S}_j(\delta))$, $j=1,\dots,k$, where $0<\delta\leq\varepsilon_1$;
- $p(\mathsf{S}_j(\delta))\subsetneq W_j$, $j=1,\dots,k$, where $0<\delta\leq\varepsilon_2$;
- $W_\mu\cap W_\nu=\{0\}$ $\forall\mu\neq\nu$, because the aperture of each W_j, $\pi/kM<2\pi/k$; and
- $p^{-1}\{0\}\cap\{\,\cup_{j=1}^k\mathsf{S}_j(\delta)\,\}=\{(0,0)\}$, where $0<\delta\leq\varepsilon_2$.

We define $\varepsilon_0:=\min(\varepsilon_1,\varepsilon_2)$; the above facts allow us to apply Lemma 2.1 repeatedly to show that

$$\mathcal{P}(\cup_{j=1}^k\mathsf{S}_j(\varepsilon_0)) \;=\; \mathcal{C}(\cup_{j=1}^k\mathsf{S}_j(\varepsilon_0)). \tag{3.5}$$

Now let $f\in\mathcal{C}(\{(z,w):|z|\leq\varepsilon_0\}\cap\mathfrak{S})$. Define $\widehat{f}:=f\circ\varPsi:\varPsi^{-1}(\{(z,w):|z|\leq\varepsilon_0\}\cap\mathfrak{S})\to\mathbb{C}$. As $\varPsi^{-1}(\{(z,w):|z|\leq\varepsilon_0\}\cap\mathfrak{S})=\cup_{j=1}^k\mathsf{S}_j(\varepsilon_0)$, $\widehat{f}\in\mathcal{C}(\cup_{j=1}^k\mathsf{S}_j(\varepsilon_0))$. We can paraphrase (3.5) in the following way : for each $\epsilon>0$, there exists a polynomial g_ϵ such that

$$|\widehat{f}(z,e^{2\pi i(j-1)/k}w)-g_\epsilon(z,e^{2\pi i(j-1)/k}w)| \;<\; \epsilon \quad \forall(z,w)\in\mathsf{S}_1(\varepsilon_0), \quad j=1,\dots,k. \tag{3.6}$$

We define

$$Q_\epsilon(z,w) \;:=\; \frac{1}{k}\sum_{j=1}^k g_\epsilon(z,e^{2\pi i(j-1)/k}w).$$

Notice that if $g_\epsilon(z,w)=\sum_{0\leq\mu+\nu\leq N}A_{\mu,\nu}z^\mu w^\nu$, then $Q_\epsilon(z,w)$ has the form

$$\begin{aligned}Q_\epsilon(z,w) &= \sum_{(\mu,\nu):\nu=kj} A_{\mu,kj}z^\mu w^{kj}\\ &\equiv P_\epsilon(z,w^k),\end{aligned}$$

where P_ϵ is itself a polynomial. Let us write $w=|w|e^{i\theta}$, $\theta\in[0,2\pi)$. For $(z,w)\in\{(z,w):|z|\leq\varepsilon_0\}\cap\mathfrak{S}$, we compute

$$\begin{aligned}|f(z,w)-P_\epsilon(z,w)| &= \left|\frac{1}{k}\sum_{j=1}^k\widehat{f}(z,|w|^{1/k}e^{i(2\pi(j-1)+\theta)/k})-Q_\epsilon(z,|w|^{1/k}e^{i\theta/k})\right|\\ &\leq \sum_{j=1}^k\frac{|\widehat{f}(z,|w|^{1/k}e^{i(2\pi(j-1)+\theta)/k})-g_\epsilon(z,|w|^{1/k}e^{i(2\pi(j-1)+\theta)/k})|}{k} \;<\; k\left(\frac{\epsilon}{k}\right).\end{aligned}$$

The last inequality follows from the estimate (3.6). This establishes that $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$.

We now only need to show that $\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ is polynomially convex. This follows from general abstract considerations. For this purpose, given a compact $K \Subset \mathbb{C}^n$, we define

$$\begin{aligned} \widehat{K} \ &:= \ \text{the polynomially convex hull of } K, \\ \mathcal{A}(K;\mathbb{C}^n) \ &:= \ \text{the uniform algebra generated by the class } \{f|_K : f \in \mathcal{O}(\mathbb{C}^n)\}, \\ \mathcal{M}[\mathcal{A}(K;\mathbb{C}^n)] \ &:= \ \text{the maximal ideal space of the uniform algebra } \mathcal{A}(K;\mathbb{C}^n). \end{aligned}$$

We know that $\mathcal{M}[\mathcal{A}(K;\mathbb{C}^n)] = \widehat{K}$; see, for instance, [4, Corollary VII.A(6)]. Thus, in our situation, $\mathcal{M}[\mathcal{A}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S};\mathbb{C}^2)] = \widehat{\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}}$. But since $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$,

$$\widehat{\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}} \ = \ \mathcal{M}[\mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})] \ = \ \{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}.$$

This concludes our proof. ☐

Before proving Theorem 1.2 we remark that with respect to a new system of holomorphic coordinates (z^*, w^*) defined by

$$\begin{aligned} z^* \ &:= \ e^{-i\phi_0} z \\ w^* \ &:= \ \frac{w - (C_{2k,0} - C_{0,2k}) z^{2k}}{C_{k,k}}, \\ \text{where } \phi_0 \ &:= \ \frac{\operatorname{Arg}(C_{0,2k}/C_{k,k})}{2k} \ , \end{aligned}$$

$\mathfrak{S}$ is expressed by the equation

$$\mathfrak{S} \ : \quad w^* \ = \ |z^*|^{2k} + \gamma((z^*)^{2k} + (\overline{z}^*)^{2k}) + \frac{\widetilde{\Sigma}(e^{i\phi_0} z^*)}{C_{k,k}} + \frac{\widetilde{F}(e^{i\phi_0} z^*)}{C_{k,k}}.$$

For simplicity of notation, we will denote the new coordinates by (z,w), and assert that the surface $\mathfrak{S}$ is expressed by an equation of the form

$$\mathfrak{S} \ : \quad w \ = \ |z|^{2k} + \gamma(z^{2k} + \overline{z}^{2k}) + \Sigma(z) + F(z) \tag{3.7}$$

with respect to these new coordinates. In the above expression, $\Sigma(z)$ is homogeneous of degree $2k$, $F(z) = o(|z|^{2k})$ as $z \to 0$, and γ is precisely as defined in Theorem 1.2. Let us define a function $\psi(z,w) := |z|^{2k} + \gamma(z^{2k} + \overline{z}^{2k}) + \Sigma(z) + F(z) - w$. With respect to the new coordinates, $\mathfrak{S}$ is thus the zero set of ψ. Under this change of coordinate $|\Sigma(z)| \leq \kappa(2\gamma - 1)|z|^{2k}$, and the estimate (1.2) transforms to

$$0 \ \leq \ \kappa \ < \frac{1}{2} \ \min \left\{ \frac{\pi}{2k}, \frac{2\gamma - 1}{2\gamma(3\gamma + 2)} \ \right\}. \tag{3.8}$$

In the remainder of this paper, we will assume that $\mathfrak{S}$ is the zero set of ψ or, equivalently, is defined by (3.7)

The proof of Theorem 1.2 : Let $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\Phi(z,w) = (z, z^k w + \gamma(z^{2k} + w^2)$. This is a proper mapping of $\mathbb{C}^2$ onto itself having multiplicity 2. We first show that there is a small constant $\delta > 0$ such that $\Phi^{-1}(\{(z,w) : |z| \leq \delta\} \cap \mathfrak{S}) = \mathcal{S}_1(\delta) \cup \mathcal{S}_2(\delta)$, where $\mathcal{S}_j(\delta),\ j = 1,2$, are compact sets given by the the following equations

$$\begin{aligned} \mathcal{S}_1(\delta) \ &: \quad w \ = \ \overline{z}^k + \mathcal{H}(z) + f_1(z), \quad |z| \leq \delta, \\ \mathcal{S}_2(\delta) \ &: \quad w \ = \ -\left(\frac{1}{\gamma} z^k + \overline{z}^k\right) - \mathcal{H}(z) + f_2(z), \quad |z| \leq \delta, \end{aligned} \tag{3.9}$$

where

- f_1 and f_2 are continuous functions satisfying $f_j(z) = o(|z|^k)$ as $z \to 0,\ j = 1,2$; and
- $\mathcal{H}(z)$ is a continuous function that is homogeneous of degree k and satisfies $|\mathcal{H}(z)| \leq 2\kappa|z|^k$.

In order to justify the above statement, we need to analyze how the equations describing $\mathcal{S}_1(\delta)$ and $\mathcal{S}_2(\delta)$ arise. We first study the set $\mathcal{S}_1(\delta)$. For this purpose, we introduce the quantity $g_1(z)$ such that

$$\mathcal{S}_1(\delta) \ = \{(z,w) : w = \overline{z}^k + g_1(z),\ |z| \text{ small}\},$$

and demand that $\psi[\Phi(z, \overline{z}^k + g_1(z))] = 0$. Thus g_1 satisfies the quadratic equation

$$\gamma g_1^2 + (2\gamma\overline{z}^k + z^k)g_1 - \{\Sigma(z) + F(z)\} \ = \ 0.$$

By the quadratic formula

$$\begin{aligned} g_1(z) \ &= \ -\frac{2\gamma\overline{z}^k + z^k}{2\gamma} + \frac{\sqrt{(2\gamma\overline{z}^k + z^k)^2 + 4\gamma\{\Sigma(z) + F(z)\}}}{2\gamma} \\ &= \ -\frac{2\gamma\overline{z}^k + z^k}{2\gamma} + \frac{2\gamma\overline{z}^k + z^k}{2\gamma}\left\{1 + \frac{4\gamma\{\Sigma(z) + F(z)\}}{(2\gamma\overline{z}^k + z^k)^2}\right\}^{1/2} \end{aligned} \tag{3.10}$$

in a small neighbourhood of $z = 0$, where the square root is unambiguously defined. We choose the positive square root in the equation (3.10) because it is this branch of the square root which ensures that $f_1(z)$ decays in the desired manner as $z \to 0$. To see this, observe that if we write $z = |z|e^{i\theta}$, then $2\gamma\overline{z}^{2k} + |z|^{2k} = |z|^{2k}\{(2\gamma\cos(2k\theta) + 1) - 2i\gamma\sin(2k\theta)\}$. Therefore,

$$|2\gamma\overline{z}^{2k} + |z|^{2k}| \ = \ |z|^{2k}\sqrt{1 + 4\gamma^2 + 4\gamma\cos(2k\theta)} \ \geq \ |z|^{2k}\sqrt{1 + 4\gamma(\gamma - 1)} \ = \ (2\gamma - 1)|z|^{2k}. \tag{3.11}$$

Since $\gamma > 1/2$, the quantity on the extreme right of the above estimate is strictly positive when $z \neq 0$. Thus :

$$\left|\frac{4\gamma\Sigma(z)}{(2\gamma\overline{z}^k + z^k)^2}\right| \ = \ \left|\frac{4\gamma\Sigma(z)\overline{z}^{2k}}{(2\gamma\overline{z}^{2k} + |z|^{2k})^2}\right| \ \leq \ \frac{4\kappa\gamma(2\gamma - 1)|z|^{2k}}{(2\gamma - 1)^2|z|^{2k}} \ \leq \ \frac{2}{7}. \tag{3.12}$$

The last inequality is a consequence of the estimate (3.8) for κ. As $|F(z)| = o(|z|^{2k})$ as $z \to 0$, there is a $\delta > 0$ which is so small that $|4\gamma\{\Sigma(z) + F(z)\}/(2\gamma\overline{z}^k + z^k)^2| < 1\ \forall |z| \leq \delta$. Therefore, we can write :

$$g_1(z) \ = \ \frac{2\gamma\overline{z}^k + z^k}{2\gamma}\sum_{m=1}^{\infty}\beta_m\left\{\frac{4\gamma\Sigma(z)}{(2\gamma\overline{z}^k + z^k)^2}\right\}^m + \frac{F(z)}{2\gamma\overline{z}^k + z^k} + O(|z|^{k+1}) \quad \forall |z| \leq \delta, \tag{3.13}$$

where the coefficients β_m are the coefficients occuring in the Taylor expansion of $(1+x)^{1/2}$ around $x=0$. The smallness of the quantity on the extreme left of (3.12) allows us to make the following estimate

$$\sum_{m=1}^{\infty}|\beta_m|\left|\frac{4\gamma\Sigma(z)}{(2\gamma\overline{z}^k+z^k)^2}\right|^m \;=\; 1-\left\{1-\left|\frac{4\gamma\Sigma(z)}{(2\gamma\overline{z}^k+z^k)^2}\right|\right\}^{1/2} \;<\; \left|\frac{4\gamma\Sigma(z)}{(2\gamma\overline{z}^k+z^k)^2}\right|. \tag{3.14}$$

From the inequalities (3.13), (3.12) and (3.14), we can write

$$\begin{aligned} g_1(z) \;&\equiv\; \mathcal{H}(z)+\frac{F(z)}{2\gamma\overline{z}^k+z^k}+O(|z|^{k+1}) \\ &\equiv\; \mathcal{H}(z)+f_1(z) \quad \forall|z|\le\delta, \end{aligned}$$

provided $\delta>0$ is sufficiently small, and where

$$\begin{aligned} \mathcal{H}(z) \;&:=\; \frac{2\gamma\overline{z}^k+z^k}{2\gamma}\sum_{m=1}^{\infty}\beta_m\left\{\frac{4\gamma\Sigma(z)}{(2\gamma\overline{z}^k+z^k)^2}\right\}^m, \\ |\mathcal{H}(z)| \;&\le\; 2\left|\frac{\Sigma(z)}{(2\gamma\overline{z}^k+z^k)}\right| \;\le\; 2\kappa|z|^k. \end{aligned}$$

By this last estimate, we see that $\mathcal{H}(z)$ and $f_1(z)$ satisfy the desired properties. This completes the analysis of the compact $\mathcal{S}_1(\delta)$.

Next, we study the set $\mathcal{S}_2(\delta)$. Let $g_2(z)$ such that

$$\mathcal{S}_2(\delta) \;=\left\{(z,w): w=-\left(\frac{z^k}{\gamma}+\overline{z}^k\right)+g_2(z),\ |z| \text{ small}\right\}.$$

Once more, we demand that $\psi[\Phi(z,\overline{z}^k+g_2(z))]=0$. Thus g_2 satisfies the quadratic equation

$$\gamma g_2^2-(2\gamma\overline{z}^k+z^k)g_2-\{\Sigma(z)+F(z)\} \;=\; 0.$$

By the quadratic formula

$$\begin{aligned} g_2(z) \;&=\; \frac{2\gamma\overline{z}^k+z^k}{2\gamma}-\frac{\sqrt{(2\gamma\overline{z}^k+z^k)^2+4\gamma\{\Sigma(z)+F(z)\}}}{2\gamma} \\ &=\; \frac{2\gamma\overline{z}^k+z^k}{2\gamma}-\frac{2\gamma\overline{z}^k+z^k}{2\gamma}\left\{1+\frac{4\gamma\{\Sigma(z)+F(z)\}}{(2\gamma\overline{z}^k+z^k)^2}\right\}^{1/2} \end{aligned} \tag{3.15}$$

for each $|z|\le\delta$, with $\delta>0$ appropriately small. Unlike in equation (3.10), we choose a different branch of the square root in (3.15). We make this choice because it ensures that $f_2(z)$ decays in the desired manner as $z\to 0$. This is shown in exactly the same manner as in the case of $\mathcal{S}_1(\delta)$. In this case, it turns out that

$$\begin{aligned} g_2(z) \;&=\; -\frac{2\gamma\overline{z}^k+z^k}{2\gamma}\sum_{m=1}^{\infty}\beta_m\left\{\frac{4\gamma\Sigma(z)}{(2\gamma\overline{z}^k+z^k)^2}\right\}^m-\frac{F(z)}{2\gamma\overline{z}^k+z^k}+O(|z|^{k+1}) \\ &\equiv\; -\mathcal{H}(z)+f_2(z) \quad \forall|z|\le\delta, \end{aligned} \tag{3.16}$$

and exactly the same $\delta>0$ as the δ produced in the analysis on $\mathcal{S}_1(\delta)$ works. Exactly as in the preceding paragraph, from the expressions (3.13), (3.12) and (3.16), the desired conclusion about the structure of $\mathcal{S}_2$ is established.

Finally, owing to the estimate $|\mathcal{H}(z)| \leq 2\kappa|z|^k$, whereby

$$2|\mathcal{H}(z)| \ < \ \left(2 - \frac{1}{\gamma}\right)|z|^k,$$

and the fact that $\gamma > 1/2$, we have

$$\overline{z}^k + \mathcal{H}(z) \ \neq \ -\left(\frac{1}{\gamma}z^k + \overline{z}^k\right) - \mathcal{H}(z) \quad \forall z \neq 0.$$

Since $f_1(z), f_2(z) = o(|z|^k)$ as $z \to 0$, the above inequality implies – lowering the value of $\delta > 0$ if necessary – that $\mathcal{S}_1(\delta) \cap \mathcal{S}_2(\delta) = \{0\}$. From this we conclude, since $\varPhi$ is a mapping of multiplicity 2 and $\varPhi[\mathcal{S}_j(\delta)] \subset \mathfrak{S},\ j = 1, 2$, that $\varPhi^{-1}(\{(z,w) : |z| \leq \delta\} \cap \mathfrak{S}) = \mathcal{S}_1(\delta) \cup \mathcal{S}_2(\delta)$.

We shall need the following lemma, whose proof is deferred to §4 :

Lemma 4.1. *Let $\mathcal{S}$ be a surface in $\mathbb{C}^2$ that, in a neighbourhood of the origin, is defined by the equation*

$$w \ = \ \overline{z}^k + \sigma(z) + G(z),$$

where

- $\gamma > 1/2$,
- *σ is a continuous function that is homogeneous of degree k such that $|\sigma(z)| \leq 2\kappa|z|^k$ for some κ satisfying the condition* (3.8)*, and*
- *G is a continuous function satisfying $G(z) = o(|z|^k)$ as $z \to 0$.*

There exists a small constant $\varepsilon_0 > 0$ such that $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S})$.

By a *direct* application of Lemma 4.1, we see that there exists an $\varepsilon_1 > 0$ such that $\mathcal{P}(\mathcal{S}_1(\varepsilon)) = \mathcal{C}(\mathcal{S}_1(\varepsilon))\ \forall \varepsilon \leq \varepsilon_1$. The image of $\mathcal{S}_2(\delta)$ under the biholomorphic map $(z,w) \mapsto (z, -w - z^k/\gamma)$ is of the same form as $\mathcal{S}$ in Lemma 4.1. Thus, $\mathcal{P}(\mathcal{S}_2(\varepsilon)) = \mathcal{C}(\mathcal{S}_2(\varepsilon))\ \forall \varepsilon \leq \varepsilon_1$.

Let $\phi_{\epsilon(\gamma)}$ denote the polynomial

$$\phi_{\epsilon(\gamma)}(z,w) = \frac{z^{2k} - w^2}{4} + \epsilon(\gamma) z^k w, \tag{3.17}$$

where

$$\epsilon(\gamma) \ := \ \begin{cases} 3/16, & \text{if } \gamma \geq 1, \\ \min\left\{\dfrac{3}{16}, \dfrac{2\gamma - 1}{8\gamma(1-\gamma)}\right\}, & \text{if } 1/2 < \gamma < 1. \end{cases}$$

We now refer the reader to Lemmas 4.2 and 4.3 in the next section. These lemmas tell us that there exists a small constant $\varepsilon_2 > 0$ such that for every $\delta < \varepsilon_2$, $\phi_{\epsilon(\gamma)}$ maps $\mathcal{S}_1(\delta)$ into a closed sector that is symmetric with respect to the x-axis and is strictly contained in $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$, while $\phi_{\epsilon(\gamma)}(\mathcal{S}_2(\delta)) \setminus \{0\}$ is contained in $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.

Let $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. At this stage in the proof, we have all the elements needed to invoke Kallin's Lemma, (i.e. Lemma 2.1) to conclude that $\mathcal{P}(\mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0)) = \mathcal{C}(\mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0))$. We want to deduce from this that $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$. This is achieved by the following argument by Forstneric & Stout given in [3]; we present it here for the reader's convenience. Let $f \in \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$. Note that as $\varPhi^{-1}(\{(z,w) : |z| \leq$

$\varepsilon_0\} \cap \mathfrak{S}) = \mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0)$, $f \circ \Phi \in \mathcal{C}(\mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0))$. From what we have shown, there is a sequence of polynimials $\{Q_n\}_{n\in\mathbb{N}}$ such that $Q_n \longrightarrow f \circ \Phi$ uniformly on $\mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0)$. Let U be the open, dense set in $\mathbb{C}^2$ such that for each $\zeta \in U$, $\Phi^{-1}\{\zeta\}$ consists of two distinct points $\zeta^{(+)}$ and $\zeta^{(-)}$. By the standard theory of analytic covers the function

$$P_n(\zeta) \ := \frac{Q_n(\zeta^{(+)}) + Q_n(\zeta^{(-)})}{2}, \quad \zeta \in U,$$

which is holomorphic in U, extends to an entire function. By construction, the sequence of entire functions $\{P_n\}_{n\in\mathbb{N}}$ converges uniformly on $\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ to f. Hence $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$.

In view of this last fact, it follows that $\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ is polynomially convex. The abstract uniform-algebras argument needed for showing this is precisely the one given in the last paragraph of the proof of Theorem 1.1. ☐

4. Technical Lemmas

In this section, we prove the three technical lemmas that we used in the proof of Theorem 1.2. But we first make the following observation, easily verified, about the quantity κ arising in the condition (3.8) :

$$\kappa \ < \ \frac{1}{2}\ \min\left\{\frac{\pi}{2k},\ \frac{2\gamma-1}{2\gamma(3\gamma+2)}\right\} \implies \kappa \ < \ \frac{1}{2}\ \min\left(\frac{1}{4}, \frac{\pi}{2k}\right), \tag{4.1}$$

which we shall use in several instances below. Firstly, we prove

Lemma 4.1. *Let $\mathcal{S}$ be the surface in $\mathbb{C}^2$ that, in a neighbourhood of the origin, is defined by the equation*

$$w \ = \ \overline{z}^k + \sigma(z) + G(z),$$

where

- $\gamma > 1/2$,
- *σ is a continuous function that is homogeneous of degree k such that $|\sigma(z)| \leq 2\kappa|z|^k$ for some κ satisfying the condition (3.8), and*
- *G is a continuous function satisfying $G(z) = o(|z|^k)$ as $z \to 0$.*

There exists a small constant $\varepsilon_0 > 0$ such that $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S})$.

Proof. We closely follow the techniques used in the proof of Theorem 1.1. As before, let $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\Psi(z,w) := (z, w^k)$. We first show that there exists a $\delta > 0$ such that $\Psi^{-1}(\{(z,w) : |z| \leq \delta\} \cap \mathcal{S}) = \cup_{j=1}^k S_j(\delta)$, where $S_1(\delta), \dots, S_k(\delta)$, are compact sets given by the equations

$$S_j(\delta) \ : \quad w \ = \ e^{2\pi i(j-1)/k}\ \overline{z}\ \{1 + H(z)\}, \quad |z| \leq \delta,$$

and H satisfies certain useful size estimates. Recall from (3.11) that $|2\gamma\overline{z}^{2k} + |z|^{2k}| \geq (2\gamma - 1)|z|^{2k}$, and that – because $\gamma > 1/2$ – the quantity on the right is strictly positive when $z \neq 0$. Now, $\mathcal{S}$ is the graph of a function G^* that may be written as

$$G^*(z) \ = \ \overline{z}^k \ \left\{1+\frac{\sigma(z)}{\overline{z}^k}+\frac{G(z)}{\overline{z}^k} \ \right\}.$$

Observe that

a) Owing to the estimate for $\sigma(z)$ and the remark (4.1)

$$\left| \ \frac{\sigma(z)}{\overline{z}^k} \ \right| \ \leq \ 2\kappa \ < \ \min\left(\frac{1}{4},\frac{\pi}{2k}\right) \ \ ; \text{ and} \tag{4.2}$$

b) $\lim_{z\to 0} G(z)/\overline{z}^k = 0$.

For the above reasons, we can choose the $\delta > 0$ introduced at the beginning of this proof to be so small that

$$\left| \ \frac{\sigma(z)}{\overline{z}^k}+\frac{G(z)}{\overline{z}^k} \ \right| \ \leq \ \frac{1}{2} \quad \forall |z| \leq \delta.$$

Given this fact, $G^*(z)$ has k distinct k^{th}-roots $g_1^*(z),\dots,g_k^*(z)$ when $z \neq 0$, each $S_j(\delta)$ is the graph of g_j^*, $j = 1,\dots,k$, and

$$g_j^*(z) \ = \ e^{2\pi i(j-1)/k} \ \overline{z} \ \left[\ 1+\sum_{m=1}^{\infty}\alpha_m\left\{\frac{\sigma(z)}{\overline{z}^k}\right\}^m + o(1) \ \right], \tag{4.3}$$

where the α_m's are exactly as in the expression (3.1). The infinite series in equation (4.3) represents a function h that is homogeneous of degree 0, and we write $H(z) := h(z)+R(z)$, where $R(z) = o(1)$ as $z \to 0$. Arguing as before

$$\sup_{|z|=1}|h(z)| \ \leq \ \sum_{m=1}^{\infty}|\alpha_m|\left\{\min\left(\frac{1}{4},\frac{\pi}{2k}\right)\right\}^m \ = \ 1-\left\{1-\min\left(\frac{1}{4},\frac{\pi}{2k}\right)\right\}^{1/k} \ < \ \min\left(\frac{1}{4},\frac{\pi}{2k}\right). \tag{4.4}$$

In view of (4.3) and (4.4), it is possible to find a small constant $\varepsilon_1 > 0$ such that

$$\begin{aligned} |\overline{z}H(z)-\overline{\zeta}H(\zeta)| \ &\leq \sup_{|\xi|=1}|h(\xi)| \ |z-\zeta| + |z| \ |h(z)-h(\zeta)| + |\overline{z}R(z)-\overline{\zeta}R(\zeta)| \\ &< \ |z-\zeta| \quad \forall z \neq \zeta \ : \ |z|,|\zeta| \leq \delta, \text{ and } \forall \delta \in (0,\varepsilon_1]. \end{aligned} \tag{4.5}$$

The estimate (4.5) allows us – as in the proof of Theorem 1.1 – to use a result of Wermer [8, Theorem 1] to conclude that for each $S_j(\delta)$, $\mathcal{P}(S_j(\delta)) = \mathcal{C}(S_j(\delta))$, $j = 1,\dots,k$, assuming, of course, that $0 < \delta \leq \varepsilon_1$.

Consider the polynomial $q(z,w) = zw$. For any $(z,w) \in S_1(\delta)$,

$$\begin{aligned} \text{Re}\{q(z,w)\} \ &= \ |z|^2+\text{Re}\{ \ |z|^2(h(z)+R(z))\} \ \geq \ |z|^2-|z|^2|h(z)|-|z|^2|R(z)|, \\ |\text{Im}\{q(z,w)\}| \ &\leq \ |z|^2\{ \ |h(z)|+|R(z)| \ \}. \end{aligned}$$

In view of the estimate (4.4) and the fact that $R(z) = o(1)$ as $z \to 0$, we can find a small constant $\varepsilon_2 > 0$ such that

$$\begin{aligned} \text{Re}\{q(z,w)\} \ \geq \ \frac{3}{4}|z|^2, \qquad |\text{Im}\{q(z,w)\}| \ < \ \frac{\pi}{2k}|z|^2 \quad \forall |z| \leq \delta, \\ q(S_1(\delta)) \ \subsetneq \ \{x+iy \in \mathbb{C} : |y| \leq (2\pi/3k)x\}, \quad \text{where } \delta \in (0,\varepsilon_2]. \end{aligned}$$

In other words, $q(S_1(\delta))$ is a proper subset of the sector W_1 that is centered on the positive x-axis, and has an aperture of $(4\pi/3k)$. Note, therefore that $q(S_j(\delta))$ is a proper subset of the sector W_j, which is simply a copy of W_1 rotated by $(2\pi(j-1)/k)$, $j = 1, \dots, k$. Furthermore, $W_\mu \cap W_\nu = \{0\}$ $\forall \mu \neq \nu$. The details for showing that $\exists \varepsilon_0 > 0$ such that $\mathcal{P}(\cup_{j=1}^k S_j(\varepsilon_0)) = \mathcal{C}(\cup_{j=1}^k S_j(\varepsilon_0))$ are no different from those given in the proof of Theorem 1.1. We omit these details. And finally, from $\mathcal{P}(\cup_{j=1}^k S_j(\varepsilon_0)) = \mathcal{C}(\cup_{j=1}^k S_j(\varepsilon_0))$ we conclude, in exactly the same way as in Theorem 1.1, that $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S})$. ☐

Next, we prove the requisite lemmas for showing that the polynomial $\phi_{\epsilon(\gamma)} : \mathbb{C}^2 \to \mathbb{C}$, defined by equation (3.17), maps the surfaces $\mathcal{S}_1(\delta)$ and $\mathcal{S}_2(\delta)$ – encountered in the proof of Theorem 1.2 – into two sectors in $\mathbb{C}$ that intersect only at the origin.

Lemma 4.2. *Let Σ_1 be the surface in $\mathbb{C}^2$ given by the equation*

$$w \;=\; \overline{z}^k + \sigma(z) + G(z),$$

where γ, $\sigma(z)$ and $G(z)$ are as in Lemma 4.1. There exist small constants $\delta_1, R_1 > 0$ such that $\phi_{\epsilon(\gamma)}(\{(z,w) : |z| \leq R_1\} \cap \Sigma_1)$ is contained in the sector $W(\delta_1) = \{x + iy \in \mathbb{C} : |y| \leq (1/\delta_1)x\}$.

Proof. We compute, for $(z,w) \in \Sigma_1$

$$\begin{aligned}
\mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &= -\mathrm{Re}\left\{\frac{2\overline{z}^k\sigma(z)}{4} + \frac{\sigma(z)^2}{4} - \epsilon(\gamma)z^k\sigma(z)\right\} + \epsilon(\gamma)|z|^{2k} + o(|z|^{2k}),\\
\mathrm{Im}\{\phi_{\epsilon(\gamma)}(z,w)\} &= \frac{1}{2}\,\mathrm{Im}(z^{2k}) - \mathrm{Im}\left\{\frac{2\overline{z}^k\sigma(z)}{4} + \frac{\sigma(z)^2}{4} - \epsilon(\gamma)z^k\sigma(z)\right\} + o(|z|^{2k}).
\end{aligned}$$

We consider the following two cases:
Case (i) : $\gamma \geq 1$ *and* $\epsilon(\gamma) = 3/16$.

In view of the estimates on $\sigma(z)$ – including the upper bound (4.2) – and the remark (4.1), we can find a $R_1 > 0$ sufficiently small such that if $(z,w) \in \Sigma_1$, we have

$$\begin{aligned}
\mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &> \frac{3}{16}(1-2\kappa)|z|^{2k} - \left\{\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4}\right\}2\kappa|z|^{2k} + o(|z|^{2k})\\
&\geq \frac{3}{32}(1-8\kappa)|z|^{2k} \quad \forall |z| \leq R_1. \qquad (4.6)
\end{aligned}$$

Notice that as $\kappa < 1/8$, $(1-8\kappa) > 0$ in the expression above.

Case (ii) : $1/2 < \gamma < 1$ *and* $\epsilon(\gamma) = \min\{(2\gamma-1)/8\gamma(1-\gamma),\ 3/16\}$.
In this case, we first compute that $\forall (z,w) \in \Sigma_1$

$$\begin{aligned}
\mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &> \frac{2\gamma-1}{8\gamma(1-\gamma)}(1-2\kappa)|z|^{2k} - \left\{\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2}\right\}2\kappa|z|^{2k} + o(|z|^{2k})\\
&= \frac{2\gamma-1}{8\gamma(1-\gamma)}|z|^{2k} + \frac{2\kappa}{8}|z|^{2k} - 2\kappa\left\{\frac{2\gamma-1}{8\gamma(1-\gamma)} + \frac{6}{8}\right\}|z|^{2k} + o(|z|^{2k}) \qquad (4.7)
\end{aligned}$$

At this point we observe that

$$2\kappa \;\leq\; \frac{2\gamma-1}{2\gamma(3\gamma+2)} \;<\; \frac{2\gamma-1}{8\gamma-6\gamma^2-1} \quad \forall \gamma \in (1/2,1).$$

In view of this fact and the estimate (4.7), we can find a constant $R_1 > 0$ such that if $(z,w) \in \Sigma_1$

$$\begin{aligned} \mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} \ &> \ \frac{2\gamma-1}{8\gamma(1-\gamma)}|z|^{2k} + \frac{2\kappa}{8}|z|^{2k} - \frac{2\gamma-1}{8\gamma-6\gamma^2-1}\left\{\frac{2\gamma-1}{8\gamma(1-\gamma)} + \frac{6}{8}\right\}|z|^{2k} + o(|z|^{2k}) \\ &> \ \frac{\kappa}{4}|z|^{2k} + o(|z|^{2k}) \ > \ \frac{\kappa}{8}|z|^{2k} \qquad \forall |z| \leq R_1. \end{aligned} \tag{4.8}$$

It is of course obvious from the expressions above that there exists a $C > 0$ such that

$$|\mathrm{Im}\{\phi_{\epsilon(\gamma)}(z,w)\}| \ \leq \ C|z|^{2k} \quad \forall (z,w) \in \{(z,w) : |z| \leq R_1\} \cap \Sigma_1. \tag{4.9}$$

The result follows from (4.6), (4.8) and (4.9). □

Lemma 4.3. *Let Σ_2 be the surface in $\mathbb{C}^2$ given by the equation*

$$w \ = \ -\left(\frac{1}{\gamma}z^k + \overline{z}^k\right) + \sigma(z) + G(z),$$

where γ, $\sigma(z)$ and $G(z)$ are as in Lemma 4.1. There exists a constant $R_2 > 0$ such that $\phi(\{(z,w) : |z| \leq R_2\} \cap \Sigma_2) \setminus \{0\}$ is contained in the open left half-plane.

Proof. Once again, we analyze this problem into two cases.

Case (i) : $\gamma \geq 1$ *and* $\epsilon(\gamma) = 3/16$.
We compute to find that for $(z,w) \in \Sigma_2$

$$\begin{aligned} \mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} \ &\leq \ -\left\{\frac{1}{2\gamma} + \epsilon(\gamma)\right\}|z|^{2k} + \left\{\epsilon(\gamma)\left(\frac{1}{\gamma}\right) + \frac{1}{4\gamma^2}\right\}|\mathrm{Re}(z^{2k})| \\ &\quad + \frac{|z|^k|\sigma(z)|}{4}\left\{\frac{2}{\gamma} + 2 + 4\epsilon(\gamma) + \frac{|\sigma(z)|}{|z|^k}\right\} + o(|z|^{2k}) \\ &< \ -\left(\frac{1}{2\gamma} - \frac{1}{4\gamma^2}\right)|z|^{2k} - \frac{3}{16}\left(1 - \frac{1}{\gamma}\right)|z|^{2k} + \frac{|z|^k|\sigma(z)|}{4}\ \frac{2+3\gamma}{\gamma} + o(|z|^{2k}). \end{aligned}$$

The last inequality is the consequence of the estimate (4.2) and the fact that $\epsilon(\gamma) = 3/16$. Now, exploiting the bounds on $|\sigma(z)|$ and κ, we get

$$\begin{aligned} \mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} \ &< \ -\left(\frac{1}{2\gamma} - \frac{1}{4\gamma^2}\right)|z|^{2k} - \frac{3}{16}\left(1 - \frac{1}{\gamma}\right)|z|^{2k} + \frac{2\gamma-1}{8\gamma^2}|z|^{2k} + o(|z|^{2k}) \\ &< \ -\frac{3}{16}\left(1 - \frac{1}{\gamma}\right)|z|^{2k} + o(|z|^{2k}) \\ &< \ 0 \qquad \forall (z,w) \in \{(z,w) : |z| \leq R_2\} \cap \Sigma_2, \end{aligned} \tag{4.10}$$

for some $R_2 > 0$ that is sufficiently small. We can find such an R_2 because in this situation $(1 - 1/\gamma) > 0$.

Case (ii) : $1/2 < \gamma < 1$ *and* $\epsilon(\gamma) = \min\{(2\gamma-1)/8\gamma(1-\gamma),\ 3/16\}$.
Following the computation done in the earlier case, we get

$$\mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} \ < \ -\left(\frac{1}{2\gamma} - \frac{1}{4\gamma^2}\right)|z|^{2k} - \epsilon(\gamma)\left(1 - \frac{1}{\gamma}\right)|z|^{2k} + \frac{|z|^k|\sigma(z)|}{4}\ \frac{2+3\gamma}{\gamma} + o(|z|^{2k}).$$

This estimate is derived from the very first line of the estimate on $\mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\}$ under Case (i), coupled with the fact that $\epsilon(\gamma) \leq 3/16$. We now use the fact that $\epsilon(\gamma) \leq (2\gamma-1)/8\gamma(1-\gamma)$ to get

$$\begin{aligned}\mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &< -\left(\frac{1}{2\gamma}-\frac{1}{4\gamma^2}\right)|z|^{2k} - \frac{2\gamma-1}{8\gamma(1-\gamma)}\left(1-\frac{1}{\gamma}\right)|z|^{2k} \\ &\quad + \frac{|z|^k|\sigma(z)|}{4}\ \frac{2+3\gamma}{\gamma} + o(|z|^{2k}) \\ &\leq -\frac{1}{2}\left(\frac{1}{2\gamma}-\frac{1}{4\gamma^2}\right)|z|^{2k} + \frac{2\kappa}{4}\left(\frac{2+3\gamma}{\gamma}\right)|z|^{2k} + o(|z|^{2k}).\end{aligned}$$

Applying the fact that $2\kappa < (2\gamma-1)/2\gamma(3\gamma+2)$ to the last inequality, we see that we can find a constant $R_2 > 0$ such that $\mathrm{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} < 0\ \forall (z,w) \in \{(z,w) : |z| \leq R_2\} \cap \Sigma_2$.

Given the last conclusion, and the inequality (4.10), the result is established. ☐

References

[1] E. Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math J. **32** (1965), 1-21.

[2] F. Forstnerič, *Analytic disks with boundaries in a maximal real submanifold of* $\mathbb{C}^2$, Ann. Inst. Fourier **37** (1987), 1-44.

[3] F. Forstnerič and E.L. Stout, *A new class of polynomially convex sets*, Arkiv Mat. **29** (1991), 51-62.

[4] R.C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice Hall, Inc., Englewood Cliffs, NJ (1965).

[5] B. Jöricke, *Local polynomial hulls of discs near isolated parabolic points*, Indiana Univ. Math. J. **46** (1997), 789-826.

[6] E. Kallin, *Fat polynomially convex sets*, Function Algebras (Proc. Internat. Sympos. on Function Algebras, Tulane University, 1965), Scott Foresman, Chicago, IL (1966), pp. 149-152.

[7] A.G. O'Farrell and K.J.Preskenis, *Uniform approximation by polynomials in two functions*, Math. Ann. **284** (1989), 529-535.

[8] J. Wermer, *Approximation on a disk*, Math. Ann. **155** (1964), 331-333.

[9] J. Wiegerinck, *Locally polynomially convex hulls at degenerated CR singularities of surfaces in* $\mathbb{C}^2$, Indiana Univ. Math. J. **44** (1995), 897-915.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 525 EAST UNIVERSITY ANN ARBOR, MI 48109
E-mail address: `bharali@umich.edu`