ON THE GROWTH OF THE BERGMAN METRIC NEAR A POINT OF INFINITE TYPE

GAUTAM BHARALI

Abstract. We derive optimal estimates for the Bergman kernel and the Bergman metric for certain model domains in $\mathbb{C}^2$ near boundary points that are of infinite type. Being unbounded models, these domains obey certain geometric constraints — some of them necessary for a non-trivial Bergman space. However, these are *mild* constraints: unlike most earlier works on this subject, we are able to make estimates for *non-convex* pseudoconvex models as well. In fact, the domains we can analyse range from being mildly infinite-type to very flat at infinite-type boundary points.

1. Introduction

Let $\Omega \subset \mathbb{C}^2$ be a pseudoconvex domain (not necessarily bounded) having a $\mathcal{C}^\infty$-smooth boundary. Let $p \in \partial\Omega$ be a point of infinite type: i.e., for each $N \in \mathbb{Z}_+$, there exists a germ of a 1-dimensional complex-analytic variety through p whose order of contact with $\partial\Omega$ at p is at least N. If $\partial\Omega$ is not Levi-flat around p, then there exist local holomorphic coordinates $(z, w; U_p)$ centered at p such that

$$\Omega \cap U_p = \{(z,w) \in U_p : \mathsf{Im}w > F(z) + R(z, \mathsf{Re}w)\}, \tag{1.1}$$

where F is a smooth, subharmonic, non-harmonic function defined in a neighbourhood of $z = 0$ that vanishes to infinite order at $z = 0$; $R(\cdot\ ,0) \equiv 0$; and R is $O(|z||\mathsf{Re}w|, |\mathsf{Re}w|^2)$. Given the infinite order of vanishing of F at $z = 0$, many of the ideas for estimating the growth the Bergman kernel and its partial derivatives — evaluated on the diagonal — as one approaches a finite-type boundary point are no longer helpful. But some of the ideas alluded to can be useful (see, e.g., item (2) below) if the function F introduced in (1.1) is the restriction of a global subharmonic, non-harmonic function. Such a function gives us a model domain

$$\Omega_F := \{(z,w) \in \mathbb{C}^2 : \mathsf{Im}w > F(z)\}, \tag{1.2}$$

which approximates $\partial\Omega$ to infinite order along the complex-tangential directions at p. This paper studies the growth the Bergman kernel (evaluated on the diagonal) and the Bergman metric on Ω_F as one approaches $(0,0) \in \mathbb{C}^2$, with certain reasonable conditions on F so that:

- the Bergman space for Ω_F — which we denote by $A^2(\Omega_F) := \mathbb{L}^2(\Omega_F, \mathbb{C}) \cap \mathcal{O}(\Omega_F)$ — is non-trivial; and
- the problem just described is tractable despite the difficulties arising from F vanishing to infinite order at $z = 0$.

The model domains defined by (1.2) are reminiscent of the domains studied in [2] but, in fact, we shall study a much wider class of model domains than those introduced in [2]. To elaborate: the domains studied in the latter paper satisfied a condition $(*)$ — refer to [2, page 2] — which involved a technical growth condition that turns out to be unnecessary. For the domains Ω_F that we consider, in this paper F will just be a radial function. I.e., it will satisfy the condition

($\bullet$) $F(z) = F(|z|)\ \forall z \in \mathbb{C}$.

2010 *Mathematics Subject Classification.* Primary: 32A36; Secondary: 32A25, 32Q35.

Key words and phrases. Bergman–Fuchs formulas, Bergman kernel, Bergman metric, infinite type, optimal estimates.

While the condition (•) limits the sorts of domains of the form (1.2) that we wish to study, there are two reasons for restricting our attention to the case where F is radial:

(1) A recurring technique for obtaining the kind of estimates that we seek is the use of scaling: information on, say, the Bergman kernel at the unit scale is classical, while an understanding of $K_\Omega(z,w)$ as $\Omega \ni (z,w) \to (0,0)$ (where $(0,0) \in \partial\Omega$) is obtained by rescaling appropriately to unit scale: see, for instance, [5] by Diederich *et al.*, [14] and [15] by Nagel *et al.*, [13] by McNeal. These methods do not seem to yield optimal estimates, even just for model domains of the form (1.2), if F vanishes to infinite order at $z = 0$ and F behaves differently along different *real* directions in $\mathbb{C}$. The work of Kim–Lee [8] — who examine a class of *convex* domains that form a proper subclass of the class of domains we shall study — suggests strongly that our problem is more tractable if F is radial.

(2) Once we assume that F is radial and $\partial\Omega_F$ is not Levi-flat, it follows that $F(z) > 0$ $\forall z \in \mathbb{C} \setminus \{0\}$: see part (a) of Theorem 1.2 below. Then (provided one has a localisation theorem for the Bergman kernel for Ω_F) the arguments of Boas *et al.* in [3] imply that information on the growth of the Bergman kernel or the Bergman metric for Ω_F yields analogous information for Ω as one approaches p through $\Omega \cap U_p$, where U_p is as introduced by (1.1) and the pair (Ω, p) satisfies the assumptions stated prior to (1.2).

The function K_Ω introduced above is defined as follows: if, for a domain $\Omega \subset \mathbb{C}^2$, $B_\Omega : \Omega \times \Omega \longrightarrow \mathbb{C}$ denotes the Bergman kernel for Ω, then $K_\Omega(z,w) := B_\Omega\big((z,w),(z,w)\big)$. We will abbreviate K_{Ω_F} as K_F.

What enables us to so significantly weaken the condition $(*)$ in [2, page 2] to (•) above, and yet expect non-trivial results, is a localisation principle for the Bergman kernel and the Bergman metric by Chen *et al.* [4]: see Section 3 for details.

With these ingredients, we get the *optimal* expressions for the growth of the quantities considered — as the inequalities (1.3) and (1.4) below show. We briefly summarise where those inequalities hold:

(i) We get upper bounds on K_F and on the Bergman metric for Ω_F that hold in a family of approach regions for $(0,0) \in \partial\Omega_F$ comprising regions with *arbitrarily high* orders of contact with $\partial\Omega_F$ at $(0,0)$, our bounds being independent of the approach region.

(ii) There exists an $\overline{\Omega}_F$-open neighbourhood ω of $(0,0)$ such that our lower bound for K_F holds true on $\omega \cap \Omega_F$.

It is well known that, *even if* F is radial, $K_F(z,w) \gtrsim \|(z,w)\|^{-2}$ is the best that one expects (for non-tangential approach) without any additional information on F. For instance: with the additional information that $(0,0) \in \partial\Omega_F$ is of finite type, we get optimal estimates because, in this case, we can find constants $C, r > 0$, and $M \in \mathbb{Z}_+$ such that

$$\mathbb{B}^2(0;r) \cap \{(z,w) : \mathsf{Im}w > C|z|^{2M}\} \subset \Omega_F \cap \mathbb{B}^2(0;r) \\ \subset \mathbb{B}^2(0;r) \cap \{(z,w) : \mathsf{Im}w > (1/C)|z|^{2M}\}.$$

Here, we can make precise estimates by exploiting the simplicity of the prototypal function $z \longmapsto |z|^{2M}$. When F vanishes to infinite order at 0, there is no obvious notion of a prototype for F. However, F exhibits, in some sense, a "controlled infinite-order vanishing" at 0 if it satisfies the condition stated right after part (a) of Theorem 1.2. This condition is motivated by the fact that it encompasses a very large class of domains, ranging from the "mildly infinite-type" to the very flat at $(0,0)$: see the examples in Section 2. To state this condition, we need the following:

Definition 1.1. An increasing function $g : [0,R] \longrightarrow \mathbb{R}$ is said to satisfy a *doubling condition* if $g(0) = 0$ and there exists a constant $\sigma > 1$ such that

$$2g(x) \leq g(\sigma x) \ \ \forall x \in [0, R/\sigma].$$

We will call the constant $\sigma > 1$ a *doubling constant* for g.

We shall also need the following notation. Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a strictly increasing function and let $f(0) = 0$. We define the function Λ_f as

$$\Lambda_f(x) := \begin{cases} -1/\log(f(x)), & \text{if } 0 < x < f^{-1}(1), \\ 0, & \text{if } x = 0. \end{cases}$$

Theorem 1.2. *Let F be a $\mathcal{C}^\infty$-smooth subharmonic function on $\mathbb{C}$ that vanishes to infinite order at $0 \in \mathbb{C}$ and is radial (i.e., satisfies condition ($\bullet$) above). Suppose the boundary of the domain $\Omega_F := \{(z, w) \in \mathbb{C}^2 : \mathsf{Im}w > F(z)\}$ is not Levi-flat around $(0, 0)$.*

(*a*) *Let f be given by the relation $f(|z|) = F(z)$ $\forall z \in \mathbb{C}$. Then, f is a strictly increasing function on $[0, \infty)$ and $\lim_{r\to\infty} f(r) = +\infty$.*

Assume that there exists a constant $R \in \big(0, f^{-1}(1)\big)$ such that $\Lambda_f|_{[0,R]}$ satisfies a doubling condition. Then:

(*b*) *There exists a constant $C_1 > 0$ and, for each $\alpha > 0$ and $N \in \mathbb{Z}_+$, there exists a constant $r(\alpha, N) > 0$ such that*

$$(1/C_1)(\mathsf{Im}w)^{-2}\big(f^{-1}(\mathsf{Im}w)\big)^{-2} \leq K_F(z, w) \leq C_1(\mathsf{Im}w)^{-2}\big(f^{-1}(\mathsf{Im}w)\big)^{-2} \tag{1.3}$$
$$\forall (z, w) \in \mathscr{A}_{\alpha,N} \cap \{(z, w) : \mathsf{Im}w < r(\alpha, N)\},$$

where $\mathscr{A}_{\alpha,N}$ denotes the approach region

$$\mathscr{A}_{\alpha,N} := \big\{(z, w) \in \Omega_F : \sqrt{|z|^2 + |\mathsf{Re}w|^2} < \alpha(\mathsf{Im}w)^{1/N}\big\}.$$

(*c*) *Furthermore, there exists a constant $r_0 > 0$ (independent of all the parameters above) such that the lower bound in* (1.3) *holds for all $(z, w) \in \Omega_F \cap \{(z, w) : \mathsf{Im}w < r_0\}$.*

A further piece of notation: we shall abbreviate $ds^2_{\Omega_F}(p; \xi, \xi)$ — i.e., the Bergman metric for Ω_F at (p, ξ), which gives the **square** of the Bergman norm of $\xi \in T^{1,0}_p\Omega_F$ — as $ds^2_F(p; \xi)$. Our next theorem provides estimates for the Bergman metric of Ω_F as one approaches $(0, 0) \in \Omega_F$.

Theorem 1.3. *Let Ω_F be the domain in $\mathbb{C}^2$ described by Theorem 1.2. Identify $T^{1,0}\Omega_F$ with $\Omega_F \times \mathbb{C}^2$ via the identification $\xi = \xi_1\big(\partial/\partial z|_p\big) + \xi_2\big(\partial/\partial w|_p\big) \leftrightarrow (p; \xi_1, \xi_2) \in \Omega_F \times \mathbb{C}^2$. Then, there exists a constant $C_2 > 0$ and, for each $\alpha > 0$ and $N \in \mathbb{Z}_+$, there exists a constant $\tau(\alpha, N) > 0$ such that*

$$\begin{aligned}(1/C_2)\Big(\big(f^{-1}(\mathsf{Im}w)\big)^{-2}|\xi_1|^2 + |\mathsf{Im}w|^{-2}|\xi_2|^2\Big) &\leq ds^2_F(z, w; \xi) \\ &\leq C_2\Big(\big(f^{-1}(\mathsf{Im}w)\big)^{-2}|\xi_1|^2 + |\mathsf{Im}w|^{-2}|\xi_2|^2\Big) \qquad (1.4) \\ \forall (z, w; \xi) \in \big(\mathscr{A}_{\alpha,N} &\cap \{(z, w) : \mathsf{Im}w < \tau(\alpha, N)\}\big) \times \mathbb{C}^2,\end{aligned}$$

where $\mathscr{A}_{\alpha,N}$ is the approach region introduced in Theorem 1.2.

We emphasise: what makes optimal estimates in the infinite-type case — even with the simplifying assumption ($\bullet$) — challenging is that there is no obvious prototype that describes the behaviour of the function F at $0 \in \mathbb{C}$. In the finite-type case, the "right" prototype for the F in (1.1) (and how this prototype changes as the point p varies) is dictated by Taylor's theorem: this is the basis of the diverse estimates derived in the papers cited above. In contrast, due to the challenge just mentioned, there are very few works in the infinite-type case: see, for instance, [8, 2, 12]. The set-up in [8, 2] is the closest to that of Theorems 1.2 and 1.3. Theorem 1.2 subsumes the main result in [2]. This is because (along with the features of F already discussed) our doubling condition on Λ_f is more permissive than the control on Λ_f required in [2] (see Section 3.3 for details). In [8], the domains Ω_F are required to satisfy the following conditions (with f, as in Theorem 1.2, such that $f(|z|) = F(z)$ $\forall z \in \mathbb{C}$):

- $f''(x) > 0$ $\forall x > 0$; and

- Λ_f extends smoothly to $x = 0$ and vanishes to finite order at 0.

The second condition does not allow Kim–Lee to study in [8] such Ω_F that are either "mildly infinite-type" or very flat at $(0, 0)$. The conditions stated in Theorem 1.2 *do* allow us to analyse Ω_F of the latter kind: an assertion that will be clearer through the examples in Section 2.

Let us recall what is meant by vanishing to infinite order at 0. In the context of the domains Ω_F, we mean that the function f is of class $\mathcal{C}^\infty([0,\infty))$, and $f^{(n)}(0) = 0$, $\lim_{x\to 0^+} f(x)/x^n = 0$ for every $n \in \mathbb{N}$.

A few analytic and geometric preliminaries are needed before the proofs of our main theorems can be given. It might be helpful to get a sense of the key ideas of our proof. A discussion of our method, plus the role of the localisation principle in [4] mentioned above, are presented in Section 3. Section 4 is devoted to essential quantitative lemmas. The proofs of the main results will be presented in Sections 5 and 6.

2. Examples

This section is devoted to presenting examples of domains of the form Ω_F that satisfy the conditions of Theorems 1.2 and 1.3. They are such that the point $(0, 0) \in \partial\Omega_F$ is a point of infinite type, but $\partial\Omega_F$ will — as we shall see — be flat to varying degrees in these examples.

Let F and f be as in Theorem 1.2. Since F is assumed to be radial and subharmonic, it is useful to recall the expression for the Laplacian on $\mathbb{C}$ in polar coordinates:

$$\triangle := \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

where we write $z = re^{i\theta}$. In view of the assumption $F(re^{i\theta}) = f(r)\ \forall r > 0$ and $\forall \theta \in \mathbb{R}$, we immediately have the following:

Lemma 2.1. *Let $F : \mathbb{C} \longrightarrow \mathbb{R}$ be a radial function and let $f : [0,\infty) \longrightarrow \mathbb{R}$ be such that $F(re^{i\theta}) = f(r)\ \forall r \geq 0$ and $\forall \theta \in \mathbb{R}$, where $f \in \mathcal{C}^2\big([0,\infty)\big)$. Suppose $f^{(n)}(x) = o(x^{2-n})$ as $x \to 0^+$ for $n = 1, 2$. Furthermore, if*

$$f''(x) + x^{-1}f'(x) \geq 0 \ \ \forall x > 0,$$

then F is subharmonic on $\mathbb{C}$.

Our first example features the familiar functions $f(x) = e^{-1/x^p}$, $x > 0$ (where $p > 0$), which vanish to infinite order at $x = 0$.

Example 2.2. *A class of domains Ω_F satisfying the conditions of Theorems 1.2 and 1.3, which includes domains for which $\partial\Omega_F$ is mildly infinite-type at $(0, 0)$.*

Consider the function $f : [0,\infty) \longrightarrow [0,\infty)$ described by the following conditions:

(a) Fixing a constant $p > 0$,

$$f(x) := \begin{cases} e^{-1/x^p}, & \text{if } 0 < x < 1/2, \\ 0, & \text{if } x = 0. \end{cases}$$

(b) $f|_{[1/2,\infty)}$ is so defined that $f|_{(0,\infty)}$ is of class $\mathcal{C}^\infty$ and strictly increasing, the function $F : \mathbb{C} \longrightarrow [0,\infty)$ given by $F(z) := f(|z|)$ is subharmonic on $\mathbb{C}$, and $\lim_{x\to\infty} f(x) = +\infty$.

We shall soon see why it is possible to satisfy all of the conditions listed in (b). But first: notice that if $p \notin \mathbb{Z}_+$, then Λ_f does not extend smoothly to $x = 0$ — which places Example 2.2 outside the realm considered by Kim–Lee in [8]. We shall see that the conditions of Theorems 1.2 and 1.3 are satisfied for p arbitrarily close to 0. With f as above, when $p \ll 1$ we say that $\partial\Omega_F$ is *mildly infinite-type at* $(0, 0)$.

Let us write

$$\phi_p(x) := e^{-1/x^p} \ \ \forall x \in (0,1).$$

We compute:

$$\phi_p'(x) = px^{-(p+1)}e^{-1/x^p}\ \big(> 0 \ \ \forall x \in (0,1)\big),$$
$$\phi_p''(x) = e^{-1/x^p} \cdot \big(p^2x^{-2(p+1)} - p(p+1)x^{-(p+2)}\big).$$

Clearly

$$\phi_p''(x) + x^{-1}\phi_p'(x) > 0 \ \ \forall x : 0 < x < 1. \tag{2.1}$$

It is well-known (we shall skip calculating further higher-order derivatives) that ϕ_p extends to $[0,1)$ to belong to $\mathcal{C}^\infty([0,1))$ and vanishes to infinite order at 0. In view of (2.1) and Lemma 2.1, we conclude that the function $\Phi_p(z) := \phi_p(|z|)$ is subharmonic on the open unit disc.

It is easy to extend $\phi_p|_{(0,1/2]}$ to a $\mathcal{C}^\infty$ function on $(0,\infty)$ by matching the n-th derivative at $1/2$, of some smooth function on $[1/2,\infty)$, with $\phi_p^{(n)}(1/2)$, $n \in \mathbb{N}$. If we call this extension f and let F be as given by (b), then, as $\triangle\Phi_p$ is *strictly* positive on the circle $\{z \in \mathbb{C} : |z| = 1/2\}$ (see (2.1) above), we can also arrange for $\triangle F > 0$ on $\{z \in \mathbb{C} : |z| \geq 1/2\}$ and, indeed, for f to have all the properties stated in (b) above.

To complete the discussion of Example 2.2, we must show that Λ_f satisfies a doubling condition. Here, $\Lambda_f(x) = x^p \ \forall x \in [0,1/2]$. Hence, if we fix some $\sigma \geq 2^{1/p}\ (> 1)$, then we have

$$2\Lambda_f(x) \leq \Lambda_f(\sigma x) \ \ \forall x \in [0, 1/2\sigma].$$

Hence, Ω_F satisfies the conditions of Theorems 1.2 and 1.3. ◀

Our next example is an illustration of a domain Ω_F where $\partial\Omega_F$ may be described to be extremely flat at $(0,0)$. There are some commonalities in the methods used in [8] and in this paper, which we shall elaborate on in Section 3. The key difference between the two approaches is that Kim–Lee rely on scaling methods in [8] to complete their proofs. Although we seek slightly different conclusions from those in [8], if we were to rely on scaling methods, then we would need a non-trivial Taylor approximation of $\Lambda_f(x)$ around $x = 0$, as is the case in [8]. This is just not available for the Λ_f in Example 2.3, which is the relevance of this example.

Example 2.3. *A domain Ω_F satisfying the conditions of Theorems 1.2 and 1.3 such that $\partial\Omega_F$ is extremely flat at* $(0,0)$.

Let ψ be the function $\phi_1|_{(0,1/2)}$, where ϕ_1 is as introduced in Example 2.2. Now consider the function $f : [0,\infty) \longrightarrow [0,\infty)$ described by the following conditions:

(a') With ψ as above,

$$f(x) := \begin{cases} e^{-1/\psi(x)}, & \text{if } 0 < x < 1/2, \\ 0, & \text{if } x = 0. \end{cases}$$

(b') $f|_{[1/2,\infty)}$ is so defined that $f|_{(0,\infty)}$ is of class $\mathcal{C}^\infty$ and strictly increasing, the function $F : \mathbb{C} \longrightarrow [0,\infty)$ given by $F(z) := f(|z|)$ is subharmonic on $\mathbb{C}$, and $\lim_{x\to\infty} f(x) = +\infty$.

As in the discussion of Example 2.2, let us write

$$\phi(x) := e^{-1/\psi(x)} \ \ \forall x \in (0,1).$$

We shall omit the essentially elementary calculations showing that ϕ extends to $[0,1)$ to belong to $\mathcal{C}^\infty([0,1))$ and vanishes to infinite order at 0. Just to indicate the calculations needed: the last statement follows from the Faá di Bruno formula for the higher derivatives of the composition of two univariate functions (see [10, Chapter 1], for instance) and the fact that

$$\lim_{x\to 0^+} e^{n/x}\sqrt{e^{-1/\psi(x)}} = \lim_{x\to 0^+} e^{n/x}\exp\left(-2^{-1}e^{1/x}\right) = \lim_{y\to 0^+} \frac{e^{-1/2y}}{y^n} = 0$$

for every $n \in \mathbb{Z}_+$.

However, it is useful to calculate couple of derivatives:

$$\phi'(x) \;=\; x^{-2}e^{1/x}\exp\big(-e^{1/x}\big)\,\big(>\,0\;\;\forall x\in(0,1)\big),$$
$$\phi''(x) \;=\; \exp\big(-e^{1/x}\big)\cdot\big(x^{-4}e^{2/x}-x^{-4}e^{1/x}-2x^{-3}e^{1/x}\big).$$

Clearly

$$\phi''(x)+x^{-1}\phi'(x)\;>\;0\;\;\forall x: 0<x<1. \tag{2.2}$$

By (2.2) and Lemma 2.1, we deduce that $\Phi(z) := \phi(|z|)$ (with $\Phi(0) := 0$) is subharmonic on the open unit disc. By arguments analogous to those for Example 2.2, it is easy to extend $\phi|_{(0,1/2]}$ to a $\mathcal{C}^\infty$ function f defined on $(0,\infty)$ so that f has all the properties listed in (b').

To complete the discussion of Example 2.3, we must show that Λ_f satisfies a doubling condition. Here, $\Lambda_f(x) = \psi(x)\ \forall x \in [0,1/2]$. Fix an σ such that

$$(\sigma-1)(\log 2)^{-1}\;\geq\;1/2.$$

Then, whenever $0 < \sigma x \leq 1/2$, we have

$$\sigma x\;\leq\;(\sigma-1)(\log 2)^{-1}\;\Rightarrow\;\log 2-\frac{1}{x}\;\leq\;-\frac{1}{\sigma x},$$

which implies that $2\psi(x) \leq \psi(\sigma x)$ whenever $0 \leq \sigma x \leq 1/2$. Therefore, Ω_F satisfies the conditions of Theorems 1.2 and 1.3. ◀

3. Preliminaries

This section is devoted to introducing the key ideas underlying the proofs in this paper. To this end, we begin by introducing some of the notation that we shall frequently use.

3.1. **Common notations.** We fix the following notation.

(1) $\mathbb{D}$ will denote the open unit disc in $\mathbb{C}$ with centre at 0, while $D(a,r)$ will denote the open disc in $\mathbb{C}$ with radius $r > 0$ and centre a.
(2) For $\xi \in \mathbb{C}^2$ (or, in general, in $\mathbb{C}^n$), $\|\xi\|$ will denote the Euclidean norm. Given points $z, w \in \mathbb{C}^n$, we shall commit a mild abuse of notation by not distinguishing between points and tangent vectors, and denote the Euclidean distance between them as $\|z-w\|$.

3.2. **On the lower bounds presented in Theorems 1.2 and 1.3.** We now present an overview of how we shall derive the lower bounds given by Theorems 1.2 and 1.3, which are the non-trivial parts of these results. Implicit in both these theorems is the fact that $A^2(\Omega_F)$ is non-trivial. This, and a lot else, follows from the following localisation result. We are able to invoke this result owing to the conclusions of part (a) of Theorem 1.2.

Result 3.1 (paraphrasing [4, Lemma 3.2] by Chen–Kamimoto–Ohsawa)**.** *Let* $\Omega := \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \mathsf{Im} w > \rho(z)\}$*, where* ρ *is a non-negative plurisubharmonic function such that* $\rho(0) = 0$ *and* $\lim_{\|z\|\to\infty}\rho(z) = +\infty$*. Let* $V \Subset U$ *be two open neighbourhoods of* $0 \in \partial\Omega$*. Then, there is a constant* $\delta \equiv \delta(U,V) > 0$ *such that*

$$K_\Omega(z,w)\;\geq\;\delta K_{\Omega\cap U}(z,w)\;\;\forall (z,w)\in\Omega\cap V, \tag{3.1}$$
$$ds^2_\Omega(z,w;\,\xi)\;\geq\;\delta ds^2_{\Omega\cap U}(z,w;\,\xi)\;\;\forall (z,w;\,\xi)\in\big(\Omega\cap V\big)\times\mathbb{C}^{n+1}. \tag{3.2}$$

This localisation result allows us to obtain lower bounds for the quantities of interest by finding lower bounds for the respective quantities associated to $\Omega_F \cap \Delta$, where Δ is a well-chosen bidisc centered at $(0,0) \in \partial\Omega_F$. We shall obtain the latter lower bounds by appealing to certain extremal problems — sometimes referred to as the Bergman–Fuchs formulas — that give the values of the Bergman kernel (evaluated on the diagonal), and of the Bergman metric, for bounded domains: see [1] by Bergman (also see [6] by Fuchs).

In this paragraph, Ω will denote an arbitrary domain in $\mathbb{C}^2$ (we restrict ourselves to $\mathbb{C}^2$ to avoid having to define further notation). One of the Bergman–Fuchs formulas is:

$$K_\Omega(z,w) \;=\; \sup\left\{\frac{|\varphi(z,w)|^2}{\|\varphi\|^2_{\mathbb{L}^2(\Omega)}} : \varphi \in A^2(\Omega)\right\} \;\; \forall (z,w)\in\Omega. \tag{3.3}$$

A related formula is known for ds^2_Ω. To see this, we need the following auxiliary quantity

$$J_\Omega(z,w;\,\xi) \;:=\; \inf\Big\{\|\varphi\|^2_{\mathbb{L}^2(\Omega)} : \varphi\in A^2(\Omega),\; \varphi(z,w)=0 \text{ and } \partial_z\varphi(z,w)\xi_1+\partial_w\varphi(z,w)\xi_2 = 1\Big\},\;\; (z,w)\in\Omega,\; \xi\in\mathbb{C}^2\setminus\{0\}. \tag{3.4}$$

The Bergman–Fuchs formula for ds^2_Ω is

$$ds^2_\Omega(z,w;\,\xi) \;=\; \frac{1}{K_\Omega(z,w)\,J_\Omega(z,w;\,\xi)} \;\; \forall (z,w;\,\xi)\in\Omega\times(\mathbb{C}^2\setminus\{0\}). \tag{3.5}$$

How these formulas help in deriving the lower bounds given by Theorems 1.2 and 1.3 is summarised as follows:

- *Step 1:* We choose a suitable bidisc Δ centered at $(0,0)$ (determined just by f). To obtain a lower bound for $K_{\Omega_F\cap\Delta}(z,s+it)$, we just need to find a suitable function $\varphi_t\in A^2(\Omega_F\cap\Delta)$, $t>f(|z|)$, such that — owing to (3.3) — $\|\varphi_t\|^2_{\mathbb{L}^2(\Omega_F\cap\Delta)}$ has an upper bound that induces the lower bound in (1.3).
- *Step 2:* The latter task reduces to estimating an integral over a region in $\mathbb{R}^4$ whose boundaries are determined by f. The doubling condition is used to break up this region of integration into sub-domains on which the relevant integral is easier to estimate to sufficient precision that we get the desired upper bound.
- *Step 3:* In view of (3.4), we must to find a suitable function $\widetilde{\varphi}_t$ belonging, this time, to the class $\big\{\varphi\in A^2(\Omega_F\cap\Delta) : \varphi(z,w)=0 \text{ and } \partial_z\varphi(z,w)\xi_1+\partial_w\varphi(z,w)\xi_2=1\big\}$ in order to deduce a lower bound for $ds^2_{\Omega_F\cap\Delta}(z,s+it;\,\xi)$. In view of (3.5), we need to obtain an upper bound of a specific form for $\|\widetilde{\varphi}_t\|^2_{\mathbb{L}^2(\Omega_F\cap\Delta)}$. A procedure analogous to that described in Step 2 applies in computing the latter upper bound.

The final estimates hinted at by the above summary lead to the lower bounds that we want — i.e., for the domain Ω_F — by the use of Result 3.1.

To conclude this section, we elaborate upon some comments made in Section 1 about our condition on Λ_f in comparison to [2].

3.3. **Relation to the main result in** [2]**.** We give a justification of the assertion in Section 1 that Theorem 1.2 subsumes the main result in [2]. Given our statements on the condition $(*)$ in [2], it suffices to show that the condition imposed on Λ_f in [2] implies that $\Lambda_f|_{[0,R]}$ satisfies a doubling condition for some $R>0$. To this end, recall that for F (and the associated f) as in [2, Theorem 1], there exist constants $\varepsilon_0>0$ and $B\geq 1$ such that

$$(1/B)\chi(x) \;\leq\; \Lambda_f(x) \;\leq\; B\chi(x) \;\; \forall x\in[0,\varepsilon_0], \tag{3.6}$$

where $\chi\in\mathcal{C}([0,\varepsilon_0])$ is an increasing function such that χ^p is convex on $(0,\varepsilon_0)$ for some $p>0$. It follows that, setting $\nu:=\min\{m\in\mathbb{N} : 2^m\geq p\}$, $\chi|_{[0,2^{-(\nu+1)}\varepsilon_0]}$ satisfies a doubling condition. Write $R:=2^{-(\nu+1)}\varepsilon_0$ and let $\sigma>1$ be a doubling constant for $\chi|_{[0,R]}$. Let $N\in\mathbb{Z}_+$ be such that $2^N\geq 2B^2$. Then, by (3.6)

$$2\Lambda_f(x) \;\leq\; 2B\chi(x) \;\leq\; 2^N(1/B)\chi(x) \;\leq\; (1/B)\chi(\sigma^N x) \;\leq\; \Lambda_f(\sigma^N x) \;\; \forall x\in[0,R/\sigma^N].$$

In view of the above discussion, it follows that Theorem 1.2 subsumes the main result in [2].

4. Technical lemmas

We present some lemmas that play a supporting role in the proofs of Theorems 1.2 and 1.3.

Lemma 4.1. *Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be a strictly increasing function satisfying $f(0) = 0$. Let $\Lambda_f|_{[0,R]}$ satisfy a doubling condition for some $R \in \big(0, f^{-1}(1)\big)$. Write $G_f := \Lambda_f^{-1}$. There exist constants $T, C' > 0$ such that*

$$0 \leq G_f(2t)^{2n} - G_f(t)^{2n} \leq C' G_f(t)^{2n} \;\; \forall t \in [0,T],$$

$n = 1, 2$.

Proof. Let $\sigma > 1$ be a doubling constant for Λ_f. Write $T := \Lambda_f(R/\sigma)$. Since, by the doubling condition,

$$2\Lambda_f(x) \leq \Lambda_f(\sigma x) \;\; \forall x \in [0, R/\sigma], \tag{4.1}$$

it follows that

$$G_f\big(2\Lambda_f(x)\big) \leq \sigma x \;\; \forall x \in [0, R/\sigma].$$

This inequality holds on the interval stated since, by (4.1), $2\Lambda_f(x) \in \mathsf{dom}(G_f)$ $\forall x \in [0, R/\sigma]$. Parametrising the latter interval by $G_f : [0,T] \longrightarrow [0, R/\sigma]$, we can take $x = G_f(t)$, $t \in [0,T]$, in the last inequality to get

$$G_f(2t) \leq \sigma G_f(t) \;\; \forall t \in [0,T].$$

This implies:

$$G_f(2t) - G_f(t) \leq (\sigma - 1) G_f(t) \;\; \forall t \in [0,T]. \tag{4.2}$$

Since

$$G_f(2\,\boldsymbol{\cdot})^2 - G_f^2 = \big(G_f(2\,\boldsymbol{\cdot}) - G_f\big)^2 + 2\big(G_f(2\,\boldsymbol{\cdot}) - G_f\big)G_f\,,$$

and

$$\begin{aligned} G_f(2\,\boldsymbol{\cdot})^4 - G_f^4 = {} & \big(G_f(2\,\boldsymbol{\cdot})^2 - G_f^2\big)^2\big(G_f(2\,\boldsymbol{\cdot}) - G_f\big)^2 \\ & + 2\big(G_f(2\,\boldsymbol{\cdot})^2 - G_f^2\big)^2\big(G_f(2\,\boldsymbol{\cdot}) - G_f\big)G_f + 2\big(G_f(2\,\boldsymbol{\cdot})^2 - G_f^2\big)^2 G_f^2\,, \end{aligned}$$

we can find an appropriate constant $C' > 0$ so that the desired conclusion follows from (4.2). □

The aim of our next two lemmas is to estimate the norms of certain functions in $A^2(\Omega_F \cap \Delta)$, where Δ is an appropriately chosen bidisc, from which we shall build candidates for such functions as can be used in the argument sketched in Steps 1–3 in Section 3.

Lemma 4.2. *Let $f \in \mathcal{C}^\infty\big([0,\infty)\big)$ be a strictly increasing function that vanishes to infinite order at 0 and let Λ_f satisfy the condition stated in Lemma 4.1. Let F and Ω_F be determined by f as described in Section 1. Write $a := \min\{f^{-1}(1), 1\}$ and write $\Delta := D(0,a) \times \mathbb{D}$. There exist constants $C^*, r_0 > 0$ such that, for any $n \in \{0,1\}$, $\alpha, t > 0$, $\beta > 1$ and $z \in \mathbb{C}$, if we write*

$$\psi(\zeta, w; \alpha, \beta, n, t, z) := \frac{|z|^\alpha\, t^\beta\, \zeta^n}{(w + it)^2} \;\; \forall (\zeta, w) \in \Omega_F \cap \Delta,$$

then

$$\|\psi(\,\boldsymbol{\cdot}\,; \alpha, \beta, n, t, z)\|^2_{\mathbb{L}^2(\Omega_F \cap \Delta)} \leq C^* t^{2(\beta-1)}\big(f^{-1}(t)\big)^{2(\alpha+n+1)} \qquad \forall (z,t) : (z, it) \in \Omega_F \cap \Delta \text{ and } t < r_0. \tag{4.3}$$

Proof. Let us write $w = u + iv$ and abbreviate $\psi(\,\boldsymbol{\cdot}\,; \alpha, \beta, n, t, z)$ as ψ. We leave it to the reader to verify that we can apply Fubini's theorem wherever necessary in the following computation:

$$\|\psi\|^2_{\mathbb{L}^2(\Omega_F \cap \Delta)} = \int_{|\zeta| \leq a} \int_{-\sqrt{1-F^2(\zeta)}}^{\sqrt{1-F^2(\zeta)}} \int_{F(\zeta)}^{\sqrt{1-u^2}} \frac{|z|^{2\alpha}\, t^{2\beta}\, |\zeta|^{2n}}{|u + i(v+t)|^4} dv\, du\, dA(\zeta)$$

$$\leq |z|^{2\alpha} t^{2\beta} \int\limits_{|\zeta|\leq a} \int\limits_{F(\zeta)}^{\infty} \int\limits_{-1}^{1} (v+t)^{-4} \left(1+\left(\frac{u}{v+t}\right)^2\right)^{-2} |\zeta|^{2n}\, du\, dv\, dA(\zeta)$$

$$\leq |z|^{2\alpha} t^{2\beta} \left(\int_{\mathbb{R}} \frac{ds}{(1+s^2)^2}\right) \int\limits_{|\zeta|\leq a} \int\limits_{F(\zeta)}^{\infty} (v+t)^{-3} |\zeta|^{2n}\, dv\, dA(\zeta)$$

$$= C|z|^{2\alpha} t^{2\beta} \int_0^a \frac{r^{2n+1}}{(t+f(r))^2}\, dr\,, \tag{4.4}$$

where $C > 0$ is a universal constant.

In the remainder of this argument, $B > 0$ will denote a constant whose value is independent of the variables involved, whose actual value is not of interest, and which may change from line to line. For any $y > 0$, set $R_y := f^{-1}(y)$. Write $G_f = \Lambda_f^{-1}$. By definition, we get

$$R_{\sqrt{t}} = G_f\left(\frac{2}{\log(1/t)}\right), \quad 0 < t < 1. \tag{4.5}$$

We now break up the interval of integration of the integral in (4.4). For simplicity of notation, we shall initially consider all t such that $0 < t < f(a)^2$, to get:

$$\begin{aligned}
\int_0^a \frac{r^{2n+1}}{(t+f(r))^2}\, dr &= \left(\int_0^{R_t} + \int_{R_t}^{R_{\sqrt{t}}} + \int_{R_{\sqrt{t}}}^a \frac{r^{2n+1}}{(t+f(r))^2}\, dr\right) \\
&\leq \int_0^{R_t} \frac{r^{2n+1}}{t^2}\, dr + \left(\int_{R_t}^{R_{\sqrt{t}}} + \int_{R_{\sqrt{t}}}^a \frac{r^{2n+1}}{4tf(r)}\, dr\right) \\
&\leq B\Big(t^{-2} R_t^{2n+2} + \frac{1}{tf(R_{\sqrt{t}})}\, a^{2n+2}\Big) + \int_{R_t}^{R_{\sqrt{t}}} \frac{r^{2n+1}}{4tf(r)}\, dr \\
&\leq B\big(t^{-2} R_t^{2n+2} + t^{-3/2}\big) + \int_{R_t}^{R_{\sqrt{t}}} \frac{r^{2n+1}}{4tf(r)}\, dr \\
&\leq B\big(t^{-2} R_t^{2n+2} + t^{-3/2}\big) \\
&\qquad + \frac{t^{-2}}{4}\left[G_f\left(\frac{2}{\log(1/t)}\right)^{2n+2} - G_f\left(\frac{1}{\log(1/t)}\right)^{2n+2}\right].
\end{aligned} \tag{4.6}$$

In the above calculation, the third inequality follows from the fact that, by definition, $a \leq 1$, while the estimate for the middle integral draws upon (4.5).

We shall now apply Lemma 4.1 to the expression in brackets in (4.6). Let $T > 0$ and $C' > 0$ be as given by that lemma. At this stage, let us fix t to be in $(0, \min\{f(a)^2, e^{-1/T}\})$. By Lemma 4.1 and (4.6):

$$\begin{aligned}
\int_0^a \frac{r^{2n+1}}{(t+f(r))^2}\, dr &\leq B(t^{-2} R_t^{2n+2} + t^{-3/2}) + \frac{C'}{4} t^{-2} G_f\left(\frac{1}{\log(1/t)}\right)^{2n+2} \\
&\leq B\big(t^{-2} R_t^{2n+2} + t^{-3/2}\big).
\end{aligned} \tag{4.7}$$

By the hypothesis that f vanishes to infinite order at 0, it follows that for any $p, q > 0$,

$$\frac{\big(f^{-1}(t)\big)^p}{t^q} \longrightarrow \infty \text{ as } t \to 0^+. \tag{4.8}$$

Thus, there is a constant $c > 0$ such that

$$R_t^{2n+2} \geq t^{1/2} \quad \forall t \in (0, c).$$

Set $r_0 := \min\{f(a)^2, e^{-1/T}, c\}$. Then, from the above inequality, (4.7) and (4.4), we get

$$\|\psi\|^2_{\mathbb{L}^2(\Omega_F\cap\Delta)} \leq C\cdot B|z|^{2\alpha}\, t^{2\beta-2}\big(f^{-1}(t)\big)^{2n+2} \ \ \forall t\in(0,r_0) \text{ and } n=0,1.$$

Recall that if $(z,it)\in\Omega_F$, then $t>F(z)=f(|z|)$. From this and the previous inequality (we set $C^*:=C\cdot B$), the lemma follows. □

A part of the proof of Theorem 1.3 requires estimates for the norms of certain functions in $A^2(\Omega_F\cap\Delta)$ that are not addressed by Lemma 4.2. Thus we need:

Lemma 4.3. *Let $f\in\mathcal{C}^\infty\big([0,\infty)\big)$, $a>0$, and Ω_F, $\Delta\subset\mathbb{C}^2$ be exactly as in Lemma 4.2. There exist constants $C^*, r_0>0$ such that, for any $n\in\{0,1\}$ and $t>0$, if we write*

$$\phi(\zeta,w;n,t) := \frac{t^3w^n}{(w+it)^3} \ \ \forall(\zeta,w)\in\Omega_F\cap\Delta,$$

then

$$\|\phi(\,\boldsymbol{\cdot}\,;n,t)\|^2_{\mathbb{L}^2(\Omega_F\cap\Delta)} \leq C^*t^{2+2n}\big(f^{-1}(t)\big)^2 \ \ \forall t\in(0,r_0). \tag{4.9}$$

Proof. As in the proof of Lemma 4.2, we write $w=u+iv$ and, for $n\in\{0,1\}$, compute:

$$\begin{aligned}\|\phi(\,\boldsymbol{\cdot}\,;n,t)\|^2_{\mathbb{L}^2(\Omega_F\cap\Delta)} &= \int_{|\zeta|\leq a}\int_{-\sqrt{1-F^2(\zeta)}}^{\sqrt{1-F^2(\zeta)}}\int_{F(\zeta)}^{\sqrt{1-u^2}} \frac{t^6(u^2+v^2)^n}{|u+i(v+t)|^6}dv\,du\,dA(\zeta)\\ &\leq t^6\int\limits_{|\zeta|\leq a}\ \int\limits_{F(\zeta)}^{\infty}\int\limits_{-1}^{1} \frac{u^{2n}}{|u+i(v+t)|^6}+\frac{v^{2n}}{|u+i(v+t)|^6}\,du\,dv\,dA(\zeta)\\ &\equiv t^6(I_1+I_2). \end{aligned}\tag{4.10}$$

Next, we estimate:

$$\begin{aligned} I_1 &= \int\limits_{|\zeta|\leq a}\ \int\limits_{F(\zeta)}^{\infty}\int\limits_{-1}^{1}(v+t)^{2n-6}\frac{\big(u/(v+t)\big)^{2n}}{\Big(1+\big(u/(v+t)\big)^2\Big)^3}\,du\,dv\,dA(\zeta)\\ &\leq \left(\int_{-1}^{1}\frac{s^{2n}}{(1+s^2)^3}\,ds\right)\int_{|\zeta|\leq a}\int_{F(\zeta)}^{\infty}(v+t)^{2n-5}\,dv\,dA(\zeta)\\ &= C\int_0^a\frac{r}{(t+f(r))^{4-2n}}\,dr\,, \end{aligned}\tag{4.11}$$

and, analogously:

$$\begin{aligned} I_2 &= \int\limits_{|\zeta|\leq a}\ \int\limits_{F(\zeta)}^{\infty}\int\limits_{-1}^{1}(v+t)^{2n-6}\frac{\big(v/(v+t)\big)^{2n}}{\Big(1+\big(u/(v+t)\big)^2\Big)^3}\,du\,dv\,dA(\zeta)\\ &\leq \left(\int_{-1}^{1}\frac{ds}{(1+s^2)^3}\right)\int_{|\zeta|\leq a}\int_{F(\zeta)}^{\infty}(v+t)^{2n-5}\,dv\,dA(\zeta)\\ &= C\int_0^a\frac{r}{(t+f(r))^{4-2n}}\,dr\,, \end{aligned}\tag{4.12}$$

where, in both estimates above, $C>0$ is a constant independent of t and n.

For any $y>0$, define $R_y := f^{-1}(y)$. In what follows, $B>0$ will denote a constant whose value is independent of the variables involved, and which may change from line to line. Since the intermediate inequalities leading to (4.13) are *completely analogous* to those in the proof of Lemma 4.2, we shall be brief. From (4.11) and (4.12):

$$\begin{aligned} I_1 + I_2 &\leq B\left(\int_0^{R_t} + \int_{R_t}^{R_{\sqrt{t}}} + \int_{R_{\sqrt{t}}}^{a} \frac{r}{(t+f(r))^{4-2n}}\,dr\right) \\ &\leq B\int_0^{R_t} \frac{r}{t^{4-2n}}\,dr + B\left(\int_{R_t}^{R_{\sqrt{t}}} + \int_{R_{\sqrt{t}}}^{a} \frac{r}{4\big(tf(r)\big)^{2-n}}\,dr\right) \\ &\leq B\big(t^{2n-4}R_t^2 + t^{3(n-2)/2}\big), \qquad (4.13) \end{aligned}$$

provided $t \in (0, \min\{f(a)^2, e^{-1/T}\})$, where $T > 0$ is as given by Lemma 4.1. The justification of the last inequality is, essentially, the argument leading to the estimate (4.6) above.

Since f vanishes to infinite order at 0, we can argue exactly as in the previous proof to obtain a constant $c > 0$ so that $R_t^2 \geq t^{1-(n/2)}$ whenever $t \in (0, c)$ (recall: $n \in \{0, 1\}$). Set $r_0 := \min\{f(a)^2, e^{-1/T}, c\}$. Then, from the last inequality, (4.10) and (4.13), the estimate (4.9) follows. ☐

5. The proof of Theorem 1.2

The proof of one half of part (a) is, essentially, the proof of [2, Lemma 3.1]. We reproduce it with the aim of providing, for clarity, a few details that were tacit in [2]. Suppose there exist $r_1 < r_2$, $r_1, r_2 \in [0, \infty)$, such that $f(r_1) \geq f(r_2)$. As f is continuous, $f|_{[0,r_2]}$ attains its maximum in $[0, r_2]$ but, owing to our assumption, there exists a point $r^* \in [0, r_2)$ such that

$$f(r^*) = \max_{r\in[0,r_2]} f(r).$$

Then, as F is a radial function,

$$F(r^*) \geq F(z) \;\; \forall z \in D(0, r_2).$$

Since F is subharmonic, the Maximum Principle implies that $F|_{D(0,r_2)} \equiv 0$. But this means that the portion $\partial\Omega_F$ in $D(0, r_2) \times D(0, r_2)$ is Levi-flat, which is a contradiction. Hence f is strictly increasing. In particular, F is non-constant. Thus, by Liouville's theorem for subharmonic functions, F is unbounded. As f is strictly increasing, it follows that $\lim_{r\to\infty} f(r) = +\infty$.

Fix $\alpha > 0$ and $N \in \mathbb{Z}_+$. We shall first find a constant $r(\alpha, N) > 0$ such that the upper bound in (1.3) holds on $\mathscr{A}_{\alpha,N} \cap \{(z, w) : \mathsf{Im} w < r(\alpha, N)\}$. By part (a), f^{-1} is well-defined. Then, with G_f as in Lemma 4.1, we have the expression

$$f^{-1}(t) = G_f\left(\frac{1}{\log(1/t)}\right), \quad 0 < t < 1 \qquad (5.1)$$

(which we have tacitly used in the proof of Lemma 4.2). When $0 < \rho \leq 1/2$,

$$f^{-1}(t/2) = G_f\left(\frac{1}{\log 2 + \log(1/t)}\right) \geq G_f\left(\frac{1}{2\log(1/t)}\right) \;\; \forall t \in (0, \rho).$$

Let C' and T be as given by Lemma 4.1. By this lemma — shrinking $\rho > 0$ if necessary so that $1/\log(1/t) \in (0, T)$ whenever $t \in (0, \rho)$ — we get

$$\begin{aligned} f^{-1}(t) - f^{-1}(t/2) &\leq G_f(1/\log(t^{-1})) - G_f(1/2\log(t^{-1})) \\ &\leq C'\,G_f(1/2\log(t^{-1})) \leq C'\,f^{-1}(t/2) \;\; \forall t \in [0, \rho). \qquad (5.2) \end{aligned}$$

Write $c := (C' + 1)^{-1}$. Since $f(x)$ vanishes to infinite order at $x = 0$, there exists a constant $r(\alpha, N) > 0$ such that $r(\alpha, N) \leq \rho$ and

$$\alpha t^{1/N} < \frac{c}{2} f^{-1}(t) \;\; \forall t \in (0, r(\alpha, N)). \qquad (5.3)$$

From (5.2) and (5.3), we see that

$$|z| + \frac{c}{2} f^{-1}(t) \ < \ f^{-1}(t/2) \ \ \forall z : 0 \leq |z| < \alpha t^{1/N}, \ 0 < t < r(\alpha, N),$$

whence the bidisc

$$\triangle(z,t) \ := \ D\Big(z, \frac{c}{2} f^{-1}(t)\Big) \times D(it, t/2) \ \subset \ \Omega_F \ \ \forall (z, it) \in \mathscr{A}_{\alpha, N} \cap \{\mathsf{Im} w < r(\alpha, N)\}.$$

Observe that the translations $T_s : (z,w) \longmapsto (z, s+w)$, $s \in \mathbb{R}$, are all automorphisms of Ω_F. Thus, by the transformation rule for the Bergman kernel, and by monotonicity, we get

$$\begin{aligned} K_F(z, s+it) \ &= \ K_F(z, it) \\ &\leq \ K_{\triangle(z,t)}(z, it) \ = \ \frac{1}{\mathrm{vol}\big(\triangle(z,t)\big)} \ \ \forall (z, s+it) \in \mathscr{A}_{\alpha, N} \cap \{\mathsf{Im} w < r(\alpha, N)\}. \end{aligned} \tag{5.4}$$

The last equality follows from the fact that $\triangle(z,t)$ is a Reinhardt domain centered at (z, it). Hence, we have found a $C_1 > 0$, which is independent of the choice of α and N, such that

$$K_F(z,w) \ \leq \ C_1 (\mathsf{Im} w)^{-2} \big(f^{-1}(\mathsf{Im} w)\big)^{-2} \ \ \forall (z,w) \in \mathscr{A}_{\alpha, N} \cap \{\mathsf{Im} w < r(\alpha, N)\} \tag{5.5}$$

(here $C_1 = 16/c^2\pi^2$), which establishes one portion of part (b).

We shall now deduce the desired lower bound. Set $a := \min\{f^{-1}(1), 1\}$. In the remainder of this proof, Δ will denote the bidisc $D(0,a) \times \mathbb{D}$. Once again, we draw upon the fact that the translations $T_s : (z,w) \longmapsto (z, s+w)$, $s \in \mathbb{R}$, are automorphisms of Ω_F, whence:

$$K_F(z, s+it) \ = \ K_F(z, it) \ \ \forall (z, s+it) \in \Omega_F \tag{5.6}$$

$$\geq \ \delta K_{\Omega_F \cap \Delta}(z, it) \ \ \forall (z, it) \in \Omega_F \cap \big(\tfrac{1}{2}\Delta\big). \tag{5.7}$$

The second inequality is a consequence of Result 3.1 applied to Ω_F, taking $U = \Delta$ and $V = \frac{1}{2}\Delta$. Part (a) of the present theorem enables the use of Result 3.1.

Now, consider the functions

$$\phi_t(\zeta, w) \ := \ -4t^2/(w+it)^2 \ \ \forall (\zeta, w) \in \Omega_F \cap \Delta,$$

where $t > 0$. In the notation of Lemma 4.2, $\phi_t = -4\psi(\boldsymbol{\cdot}\,; 0, 2, 0, t, 1)$. Let $r_0 > 0$ be as given by Lemma 4.2. By construction, $\phi_t(\zeta, it) = 1 \ \forall t > 0$. Thus, by the Bergman–Fuchs identity (3.3) and the estimate (4.3) applied to ϕ_t $(= -4\psi(\boldsymbol{\cdot}\,; 0, 2, 0, t, 1)$, as explained), we have

$$K_{\Omega_F \cap \Delta}(z, it) \ \geq \ (C^*)^{-1} t^{-2} \big(f^{-1}(t)\big)^{-2} \ \ \forall (z, it) \in \Omega_F \cap \Delta \text{ and } t < r_0. \tag{5.8}$$

Lowering the value of r_0, if necessary, we may assume that $\Omega_F \cap \{(z,w) : \mathsf{Im} w < r_0\} \subseteq \Omega_F \cap \big(\frac{1}{2}\Delta\big)$. Then, from (5.6), (5.7) and (5.8), we get

$$K_F(z,w) \ \geq \ \delta (C^*)^{-1} (\mathsf{Im} w)^{-2} \big(f^{-1}(\mathsf{Im} w)\big)^{-2} \ \ \forall (z,w) \in \Omega_F \cap \{(z,w) : \mathsf{Im} w < r_0\}. \tag{5.9}$$

This establishes part (c) of our theorem. We may assume that each $r(\alpha, N) \leq r_0$ without affecting the inequality (1.3). Now, consider the constant C_1 introduced in (5.5): raising the value of C_1, if necessary, we obtain a $C_1 > 0$ such that part (b) of our theorem follows from the last observation, (5.5) and (5.9). □

6. The proof of Theorem 1.3

Part (a) of Theorem 1.2 is relevant to this proof as well. It establishes that f is invertible. Also relevant is the argument in the second paragraph of the proof of Theorem 1.2. The conclusion of this argument is summarised by the following:

Fact. *There exists a constant $c > 0$ and, for each $\alpha > 0$ and $N \in \mathbb{Z}_+$, there exists a constant $r(\alpha, N) > 0$ such that whenever $(z, it) \in \mathscr{A}_{\alpha, N} \cap \{(z,w) : \mathsf{Im} w < r(\alpha, N)\}$, the bidisc*

$$\triangle(z,t) \ := \ D\Big(z, \frac{c}{2} f^{-1}(t)\Big) \times D(it, t/2) \ \subset \ \Omega_F. \tag{6.1}$$

By an argument analogous to the one in the proof of Theorem 1.2 — involving the fact that $T_s : (z, w) \longmapsto (z, s + w)$ is an automorphism of Ω_F for any $s \in \mathbb{R}$ — we have

$$ds_F^2(z, s + it;\, \xi) \;=\; ds_F^2(z, it;\, \xi) \;\; \forall (z, s + it;\, \xi) \in \Omega_F \times \mathbb{C}^2. \tag{6.2}$$

By (6.1), and by the monotonicity property of the functional J_Ω given by (3.4), we have:

$$J_{\triangle(z,t)}(z, it;\, \xi) \;\leq\; J_{\Omega_F}(z, it;\, \xi) \;\; \forall (z, it;\, \xi) \in \big(\mathscr{A}_{\alpha,\,N} \cap \{(z, w) : \mathsf{Im} w < r(\alpha, N)\}\big) \times (\mathbb{C}^2 \setminus \{0\}).$$

We know that $K_{\triangle(z,t)}(z, it) = 16(\pi c)^{-2}\big(f^{-1}(t)\big)^{-2} t^{-2}$; see (5.4). It is a standard result (or one may compute from the last formula) that

$$ds^2_{\triangle(z,it)}(z, it;\, \xi) \;=\; 8\big(c^{-2}\big(f^{-1}(t)\big)^{-2}|\xi_1|^2 + t^{-2}|\xi_2|^2\big) \;\; \forall \xi \in \mathbb{C}^2.$$

From these formulas and (3.5), we get an *exact* expression for $J_{\triangle(z,t)}(z, it;\, \xi)$. We combine this with monotonicity of J_Ω: then, (6.2), the Bergman–Fuchs formula for ds_F^2, and the lower bound in (1.3) imply that there exists a constant $C_2 > 0$ (independent of α and N) such that:

$$\begin{aligned} ds_F^2(z, s + it;\, \xi) \;&\leq\; 1/\big(J_{\triangle(z,t)}(z, it;\, \xi)\, K_F(z, it)\big) \\ &=\; 16(\pi c)^{-2}\big(f^{-1}(t)\big)^{-2} t^{-2}\big(8c^{-2}\big(f^{-1}(t)\big)^{-2}|\xi_1|^2 + 8t^{-2}|\xi_2|^2\big)\, K_F(z, it)^{-1} \\ &\leq\; C_2\big(\big(f^{-1}(t)\big)^{-2}|\xi_1|^2 + |t|^{-2}|\xi_2|^2\big) \\ &\qquad\qquad \forall (z, s + it;\, \xi) \in \big(\mathscr{A}_{\alpha,\,N} \cap \{(z, w) : \mathsf{Im} w < r(\alpha, N)\}\big) \times (\mathbb{C}^2 \setminus \{0\}). \end{aligned}$$

Hence, we have found a $C_2 > 0$, which is independent of the choice of α and N, such that

$$\begin{aligned} ds_F^2(z, w;\, \xi) \leq\; & C_2\big(\big(f^{-1}(\mathsf{Im} w)\big)^{-2}|\xi_1|^2 + |\mathsf{Im} w|^{-2}|\xi_2|^2\big) \\ & \forall (z, w;\, \xi) \in \big(\mathscr{A}_{\alpha,\,N} \cap \{(z, w) : \mathsf{Im} w < r(\alpha, N)\}\big) \times \mathbb{C}^2, \end{aligned} \tag{6.3}$$

which establishes one half of the estimate (1.4).

We shall now deduce the desired lower bound. As in the proof of Theorem 1.2, we set $a := \min\{f^{-1}(1), 1\}$ and $\Delta := D(0, a) \times \mathbb{D}$. Also, for reasons analogous to those in the proof of Theorem 1.2 (or in the previous paragraph), we have:

$$ds_F^2(z, s + it;\, \xi) \;=\; ds_F^2(z, it;\, \xi) \;\; \forall (z, s + it;\, \xi) \in \Omega_F \times \mathbb{C}^2 \tag{6.4}$$

$$\geq\; \delta ds^2_{\Omega_F \cap \Delta}(z, it;\, \xi) \;\; \forall (z, it;\, \xi) \in \big(\Omega_F \cap \big(\tfrac{1}{2}\Delta\big)\big) \times \mathbb{C}^2. \tag{6.5}$$

The second inequality follows from Result 3.1 applied to Ω_F, taking $U = \Delta$ and $V = \frac{1}{2}\Delta$ — the applicability of this result being, as before, due to Theorem 1.2-(a).

Now, **fix** a point $(z, it) \in \Omega_F \cap \Delta$, and let $\xi \in \mathbb{C}^2 \setminus \{(0, 0)\}$. In view of (6.5), we need to find a lower bound for $ds^2_{\Omega_F \cap \Delta}(z, it;\, \xi)$. This quest for a lower bound splits into two cases. In the argument below, $B > 0$ will denote a constant whose value is independent of the variables involved, whose actual value is not of interest, and which may change from line to line.

Case 1. $\xi \in \mathbb{C}^2 \setminus \{(0, 0)\}$ *such that* $\xi_2 \neq 0$.
Consider the function

$$\phi_{t,\,\xi}(\zeta, w) \;:=\; -\frac{8it^3(w - it)}{\xi_2(w + it)^3} \;\; \forall (\zeta, w) \in \Omega_F \cap \Delta.$$

It is easy to check that $\phi_{t,\,\xi}$ belongs to the set occurring on the right-hand side of the equation that defines $J_{\Omega_F \cap \Delta}(z, it;\, \xi)$. In terms of the notation of Lemma 4.3,

$$|\phi_{t,\,\xi}|^2 \;\leq\; \frac{128}{|\xi_2|^2}\, |\phi(\boldsymbol{\cdot}\,; 1, t)|^2 + \frac{128t^2}{|\xi_2|^2}\, |\phi(\boldsymbol{\cdot}\,; 0, t)|^2. \tag{6.6}$$

Let $r_0 > 0$ be the constant given by Lemma 4.3. Let us now consider $(z, it) \in \Omega_F \cap \Delta$ such that $0 < t < r_0$. Then, in view of (6.6), Lemma 4.2 gives us

$$\|\phi_{t,\xi}\|^2_{\mathbb{L}^2(\Omega_F\cap\Delta)} \;\leq\; \frac{B}{|\xi_2|^2} t^4 \big(f^{-1}(t)\big)^2.$$

for some constant $B > 0$. Therefore, by the definition of $J_{\Omega_F\cap\Delta}(z, it;\, \xi)$ in (3.4), clearly

$$J_{\Omega_F\cap\Delta}(z, it;\, \xi) \;\leq\; \frac{B}{|\xi_2|^2}\, t^4\big(f^{-1}(t)\big)^2 \quad \forall (z, it) \in \Omega_F \cap \Delta \cap \{(z,w) : \mathsf{Im} w < r_0\} \text{ and } \xi : \xi_2 \neq 0. \tag{6.7}$$

Let us define

$$\tau(\alpha, N) \;:=\; \min\{r_0, r(\alpha, N)\} \tag{6.8}$$

where $r(\alpha, N)$ is as given by the Fact stated at the beginning of this proof. Now, the latter parameter is precisely the one provided by the proof of Theorem 1.2 and which is introduced just before the estimate (1.3). Therefore, we have, by (1.3):

$$\frac{1}{K_F(z, it)} \;\geq\; (1/C_1)t^2\big(f^{-1}(t)\big)^2 \;\; \forall (z, it) \in \mathscr{A}_{\alpha, N} \cap \{(z,w) : \mathsf{Im} w < r(\alpha, N)\}.$$

From the latter inequality, (6.7), the Bergman–Fuchs identity (3.5), and (5.7), we get

$$ds^2_{\Omega_F\cap\Delta}(z, it;\, \xi) \;\geq\; \frac{\delta|\xi_2|^2}{B\cdot C_1} t^{-2} \quad \forall (z, it) \in \mathscr{A}_{\alpha, N} \cap \big(\tfrac{1}{2}\Delta\big) \cap \{(z,w) : \mathsf{Im} w < \tau(\alpha, N)\} \text{ and } \xi : \xi_2 \neq 0. \tag{6.9}$$

Case 2. *$\xi \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $\xi_1 \neq 0$.*
Consider the function

$$\varphi_{z,t,\xi}(\zeta, w) \;:=\; -\frac{4(\zeta - z)t^2}{\xi_1(w + it)^2} \;\; \forall (\zeta, w) \in \Omega_F \cap \Delta.$$

It is easy to verify that $\varphi_{z,t,\xi}$ belongs to the set occurring on the right-hand side of the equation that defines $J_{\Omega_F\cap\Delta}(z, it;\, \xi)$. In this case, in terms of the notation of Lemma 4.2,

$$|\varphi_{z,t,\xi}|^2 \;\leq\; \frac{32}{|\xi_1|^2}\, |\psi(\cdot\,; 0, 2, 1, t, 1)|^2 + \frac{32}{|\xi_1|^2}\, |\psi(\cdot\,; 1, 2, 0, t, z)|^2. \tag{6.10}$$

As before, let us first restrict (z, it) to $\Omega_F \cap \Delta$ such that $0 < t < r_0$, where r_0 is as given by Lemma 4.2. Given (6.10), this lemma implies:

$$\|\varphi_{z,t,\xi}\|^2_{\mathbb{L}^2(\Omega_F\cap\Delta)} \;\leq\; \frac{B}{|\xi_1|^2}\, t^2\big(f^{-1}(t)\big)^4,$$

for some constant $B > 0$. Therefore, by the definition of $J_{\Omega_F\cap\Delta}(z, it;\, \xi)$ in (3.4),

$$J_{\Omega_F\cap\Delta}(z, it;\, \xi) \;\leq\; \frac{B}{|\xi_1|^2}\, t^2\big(f^{-1}(t)\big)^4 \quad \forall (z, it) \in \Omega_F \cap \Delta \cap \{(z,w) : \mathsf{Im} w < r_0\} \text{ and } \xi : \xi_1 \neq 0. \tag{6.11}$$

Defining $\tau(\alpha, N)$ exactly as in (6.8) and arguing exactly as in Case 1, we get

$$ds^2_{\Omega_F\cap\Delta}(z, it;\, \xi) \;\geq\; \frac{\delta|\xi_1|^2}{B\cdot C_1}\big(f^{-1}(t)\big)^{-2} \quad \forall (z, it) \in \mathscr{A}_{\alpha, N} \cap \big(\tfrac{1}{2}\Delta\big) \cap \{(z,w) : \mathsf{Im} w < \tau(\alpha, N)\} \text{ and } \xi : \xi_1 \neq 0. \tag{6.12}$$

To complete the proof, we first note that for each relevant (z, it), (6.9) and (6.12) give two different lower bounds for $ds^2_{\Omega_F\cap\Delta}(z, it; \boldsymbol{\cdot})$ on the set $\{\xi \in T^{1,0}_{(z,it)}(\Omega_F \cap (\frac{1}{2}\Delta)) : \xi_1 \neq 0 \text{ and } \xi_2 \neq 0\}$. From this, it follows that

$$ds^2_{\Omega_F\cap\Delta}(z, it; \xi) \geq \frac{\delta}{2B \cdot C_1}\big((f^{-1}(t))^{-2}|\xi_1|^2 + t^{-2}|\xi_2|^2\big)$$
$$\forall (z, it; \xi) \in \big(\mathscr{A}_{\alpha,N} \cap \big(\tfrac{1}{2}\Delta\big) \cap \{(z, w) : \mathsf{Im} w < \tau(\alpha, N)\}\big) \times \mathbb{C}^2. \quad (6.13)$$

Lowering the value of the constant r_0 that occurs in (6.8), if necessary, we may assume that

$$\Omega_F \cap \{(z, w) : \mathsf{Im} w < r_0\} \subseteq \Omega_F \cap \big(\tfrac{1}{2}\Delta\big).$$

Then, from (6.4), (6.5) and (6.13), we get

$$ds^2_F(z, w; \xi) \geq \frac{\delta^2}{2B \cdot C_1}\big((f^{-1}(\mathsf{Im} w))^{-2}|\xi_1|^2 + |\mathsf{Im} w|^{-2}|\xi_2|^2\big)$$
$$\forall (z, w; \xi) \in \big(\mathscr{A}_{\alpha,N} \cap \{(z, w) : \mathsf{Im} w < \tau(\alpha, N)\}\big) \times \mathbb{C}^2. \quad (6.14)$$

This establishes the other half of the estimate (1.4). Raising the value of the constant $C_2 > 0$ introduced just prior to (6.3), if necessary, (1.4) now follows from (6.3) and (6.14). □

Acknowledgements

I thank the referees of this work for helpful suggestions on exposition, and for pointing out a simpler expression for one of the conditions needed above. I specially thank the referee who drew my attention to a gap in a proof in an earlier version of this paper. This work is supported by a Swarnajayanti Fellowship (Grant No. DST/SJF/MSA-02/2013-14) and by a UGC CAS-II grant (Grant No. F.510/25/CAS-II/2018(SAP-I)).

References

[1] S. Bergman, *Über die Kernfunktion und ihr Verhälten am Rande. I*, J. Reine Angew. Math. **169** (1933), 1-42,

[2] G. Bharali, *On the growth of the Bergman kernel near an infinite-type point*, Math. Ann. **347** (2010), no. 1, 1-13.

[3] H.P. Boas, E.J. Straube, J. Yu, *Boundary limits of the Bergman kernel and metric*, Michigan Math. J. **42** (1995), no. 3, 449-461.

[4] B.-Y. Chen, J. Kamimoto and T. Ohsawa, *Behavior of the Bergman kernel at infinity*, Math. Z. **248** (2004), no. 4, 695-708.

[5] K. Diederich, G. Herbort and T. Ohsawa, *The Bergman kernel on uniformly extendable pseudoconvex domains*, Math. Ann. **273** (1986), 471-478.

[6] B. Fuchs, *Über geodätische Mannigfaltigkeiten einer bei pseudokonformen Abbildungen invarianten Reimannschen Geometrie*, Mat. Sbornik **44** (1937), 567-594.

[7] F. Haslinger, *Bergman and Hardy spaces on model domains*, Illinois J. Math. **42** (1998), no. 3, 458-469.

[8] K.T. Kim and S. Lee, *Asymptotic behavior of the Bergman kernel and associated invariants in certain infinite-type pseudoconvex domains*, Forum Math. **14** (2002), no. 5, 775-795.

[9] J.J. Kohn, *Boundary behavior of $\overline{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Diff. Geom. **6** (1972), 523-542.

[10] S.G. Krantz and H.R. Parks, A Primer of Real Analytic Functions, 2nd edition, Birkhuser Boston, Inc., Boston, 2002.

[11] S.G. Krantz and J. Yu, *On the Bergman invariant and curvatures of the Bergman metric*, Illinois J. Math. **40** (1996), no. 2, 226-244.

[12] H. Lee, *Asymptotic expansion of the Bergman kernel for tube domain of infinite type*, Bull. Korean Math. Soc. **50** (2013), no. 1, 285-303.

[13] J.D. McNeal, *Boundary behavior of the Bergman kernel function in $\mathbb{C}^2$*, Duke Math. J. **58** (1989), 499-512.

[14] A. Nagel, J.-P. Rosay, E.M. Stein, and S. Wainger, *Estimates for the Bergman and Szegö kernels in certain weakly pseudoconvex domains*, Bull. Amer. Math. Soc. (N.S.) **18** (1988), 55-59.

[15] A. Nagel, J.-P. Rosay, E.M. Stein, and S. Wainger, *Estimates for the Bergman and Szegö kernels in $\mathbb{C}^2$*, Ann. of Math. (2) **129** (1989), no. 1, 113-149.

[16] T. Ohsawa, *Boundary behavior of the Bergman kernel function on pseudoconvex domains*, Publ. Res. Inst. Math. Sci. **20** (1984), no. 4, 897-902.

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail address: bharali@iisc.ac.in