LECTURE 6 6/03/2010

* Let $\Omega \subsetneq \mathbb{C}^n$ be a domain with $\mathcal{C}^1$-smooth boundary. Given a nbhd. U of $\overline{\Omega}$, we call a function $\rho : U \to \mathbb{R}$ a defining function for Ω if $\rho \in \mathcal{C}^1$ and:
  1) $\Omega = \{z \in U : \rho(z) < 0\}$ and $\partial\Omega = \rho^{-1}\{0\}$.
  2) $\nabla\rho(\zeta) \neq 0 \ \forall \zeta \in \partial\Omega$.

It can be checked — and we shall assume this for the moment — that if Ω has $\mathcal{C}^1$-smooth boundary, then it admits a defining function.

* The aim of today's lecture is to prove the following

* Theorem 6.1. Let $\Omega \subsetneq \mathbb{C}^n$ be a domain. The following are equivalent:
  1) Ω is a domain of holomorphy
  2) For any fixed bounded, open polydisc $\Delta \subset \mathbb{C}^n$ with centre 0, $-\log d_\Omega^\Delta \in psh(\Omega)$.
  3) For any fixed bounded, open polydisc $\Delta \subset \mathbb{C}^n$ with centre 0,
  $$d_\Delta(K, \Omega^c) = d_\Delta(\widehat{K}_\Omega, \Omega^c) \ \forall\ K \subset \Omega \text{ compact}$$
  4) $\exists f \in \mathcal{O}(\Omega)$ such that it is impossible to find a pair of open sets (Ω_1, Ω_2) having the properties (i) ~ (iii) given in Definition 3.1

Furthermore, if Ω has $\mathcal{C}^2$-smooth boundary, then:
- Ω is Levi pseudoconvex

is equivalent to each of (1) — (4)

Here, we need the following definition:

Definition 6.2. Let $\Omega \subsetneq \mathbb{C}^n$ be a domain with $\mathcal{C}^2$-smooth boundary. Let ρ be a defining function for Ω. We say that Ω is Levi pseudoconvex if,

$$\left.\begin{array}{l} \text{The form } \displaystyle\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(\zeta) v_j \bar{v}_k \ \text{ is positive definite} \\ \forall\, v \in T_\zeta(\partial\Omega) \cap i T_\zeta(\partial\Omega) =: H_\zeta(\partial\Omega) \ \forall\, \zeta \in \partial\Omega \end{array}\right\} (*)$$

[The condition (*) being independent of choice of ρ.].

Scheme of proof :

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
(1) \arrow[r, Rightarrow, bend left] \arrow[d] & (2) \arrow[l, Rightarrow, bend left, "\text{VERY HARD called the ``Levi Problem''}"] \\
(5) \arrow[r, Rightarrow] & (3) \arrow[u, Rightarrow] \arrow[ul, Rightarrow]
\end{tikzcd}
$$

Check that each property is equivalent to the other. In this lecture, we shall prove

(1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (5)

* Proof of (1) $\Rightarrow$ (2)

For each $(z_0, v, r) \in \Omega \times \mathbb{C}^n \times \mathbb{R}_+$, define

$$\omega(z_0, v, \cancel{r}) := \{\zeta \in \mathbb{C} : z_0 + \zeta v \in \Omega\}$$

$$\mathbb{D}(z_0, v, r) := \{z_0 + \zeta v : \zeta \in \overline{D(0;r)}\}$$

where $r > 0$ is sufficiently small that $\mathbb{D}(z_0, v, r) \subset \Omega$.

Suppose $p \in \mathbb{C}[\zeta]$ such that

$$-\log d_\Omega^\Delta(z_0 + \zeta v) \leqslant \operatorname{Re} p(\zeta) \quad \forall\, \zeta \in \partial D(0;r)$$

Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be such that $P(z_0 + \zeta v) = p(\zeta)$. Then

$$d_\Omega^\Delta(z) \geqslant \left| e^{-P(z)} \right| \quad \forall\, z \in \partial\mathbb{D}(z_0, v, r) \qquad \text{—} \underline{(6.1)}$$

where we denote $\{z_0 + \zeta v : \zeta \in \partial D(0;r)\}$ by $\partial\mathbb{D}(z_0, v, r)$. It is easy to see that if $K := \partial\mathbb{D}(z_0, v, r)$, then $\widehat{K}_\Omega = \mathbb{D}(z_0, v, r)$. Now recall that the following can be extracted from the proof of Theorem 3.5.

If Ω is a domain of holomorphy, then for any $f \in \mathcal{O}(\Omega)$,

$$|f(z)| \leqslant d_\Omega^\Delta(z) \quad \forall\, z \in K$$

$$\Rightarrow |f(z)| \leqslant d_\Omega^\Delta(z) \quad \forall\, z \in \widehat{K}_\Omega.$$

By hypothesis and (6.1), we conclude :

For any ~~$P\in\mathbb{C}[z_1,\dots,z_n]$~~ $p\in\mathbb{C}[\zeta]$ such that
$$-\log d_\Omega^\Delta(z_0+\zeta v)\leq \operatorname{Re}p(\zeta)\ \forall\,\zeta\in\partial D(0;r),$$
we have
$$|e^{-p(z)}|\leq d_\Omega^\Delta(z)\ \forall\, z\in \mathbb{D}(z_0,v,r)$$
$$\Leftrightarrow -\log d_\Omega^\Delta(z_0+\zeta v)\leq \operatorname{Re}p(\zeta)\ \forall\,\zeta\in D(0;r)$$
$(**)$

Check that by applying suitable translations, and adjusting r if necessary, ~~we would~~ it suffices to show $(**)$ for any harmonic function $h\in\mathrm{har}(D(0;r))\cap\mathcal{C}(\overline{D(0;r)})$ replacing p there to infer that
$$\zeta\longmapsto -\log d_\Omega^\Delta(z_0+\zeta v)\ \text{is subharmonic on}\ \omega(z_0,v)$$
But now, when $h(\zeta)\geq -\log d_\Omega^\Delta(z_0+\zeta v)\ \forall\,\zeta\in\partial D(0;r)$, as h is $\mathbb{R}$-valued, we have the Fourier ~~ser~~ expansion
$$h(re^{i\theta})\sim\sum_{\nu\in\mathbb{Z}}a_\nu e^{i\nu\theta},$$
with $a_{-\nu}=\overline{a_\nu}\ \forall\,\nu\in\mathbb{N}$. Let σ_N denote the N^{th} Cesaro mean of the above series. Then
$$\sigma_N(\theta)=\operatorname{Re}\widetilde{p}_N(re^{i\theta})$$
for some polynomial $\widetilde{p}_N$. Thus, by Fejér's Theorem, for each $\nu\in\mathbb{Z}_+$, $\exists\, L(\nu)$ such that
$$\left|h(re^{i\theta})-\operatorname{Re}\widetilde{p}_\nu(re^{i\theta})\right|\leq 1/\nu\quad \forall\,\nu\geq L(\nu),\ \forall\,\theta\in[0,2\pi] \tag{6.2}$$

Thus, if we define
$$p_\nu:=\widetilde{p}_{L(\nu)}+1/\nu,$$
Then, by hypothesis:
$$-\log d_\Omega^\Delta(z_0+\zeta v)\leq \operatorname{Re}p_\nu(\zeta)\quad \forall\,\zeta\in\partial D(0;r),\ \forall\,\nu\in\mathbb{Z}_+$$
By $(**)$
$$-\log d_\Omega^\Delta(z_0+\zeta v)\leq \operatorname{Re}p_\nu(\zeta)=\operatorname{Re}\left[\frac{1}{\nu}+\widetilde{p}_{L(\nu)}(\zeta)\right]\ \forall\,\zeta\in D(0,r).$$

Taking $\nu\longrightarrow+\infty$ gives us the desired conclusion. Since $(z_0,v)\in\Omega\times\mathbb{C}^n$ were arbitrary, we have that $-\log d_\Omega^\Delta\in\mathrm{psh}(\Omega)$.

$*$ Finally, we are in a position to prove $\underline{(1)}\Rightarrow\underline{(5)}$.

Assume (5) is not true; we will use this to infer that (1) is not true. So, $\exists\, p \in \partial\Omega$ and a $V^0 \in H_p(\partial\Omega)$ such that

$$\sum_{k,j=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\, v_j^0 \overline{v}_k^0 < 0.$$

Recall the change-of-coordinate formula presented in Lecture 18. It says that — making a $\mathbb{C}$-affine change of coordinates if necessary — we may assume

* $p = 0$
* $V^0 = (1, 0, \dots, 0)$ and $H_p(\partial\Omega) = \{ z \in \mathbb{C}^n \mid z_n = 0 \}$.
* ρ is of the form
  $$\rho(z^*, z_n) = \mathrm{Re}(z_n) + h(z^*, \mathrm{Im}(z_n)) \quad \forall\, z \in \mathbb{B}^n(0;\varepsilon),$$
  where h has a critical point at $(0,0)$.

By Taylor's theorem

$$\rho(z^*, z_n) = \mathrm{Re}(z_n) + \mathrm{Re}\left[\sum_{j,k=1}^{n-1} \frac{\partial^2 h}{\partial z_j \partial z_k}(0) z_j z_k \right] + \sum_{j,k=1}^{n-1} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k + O\big(|\mathrm{Im}(z_n)|\, \|z^*\| + |\mathrm{Im}(z_n)|^2 \big) + o\big(\|z^*\|^2 \big), \quad \forall\, (z^*, z_n) \in \mathbb{B}^n(0;\varepsilon).$$

Now consider the biholomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \equiv \Phi(z)^T := \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ z_n + \sum_{j,k=1}^{n-1} \frac{\partial^2 h}{\partial z_j \partial z_k}(0) z_j z_k \end{pmatrix}$$

$\Phi(\partial\Omega \cap \mathbb{B}^n(0;\varepsilon)) = \rho \circ \big(\Phi|_{\mathbb{B}^n(0;\varepsilon)} \big)^{-1} = \widetilde{\rho}$. The effect of this transformation is that

$$(\star)\quad \widetilde{\rho}(w^*, w_n) = \mathrm{Re}(w_n) + \sum_{j,k=1}^{n-1} \frac{\partial^2 h}{\partial w_j \partial \overline{w}_k}(0)\, w_j \overline{w}_k + \text{higher-order terms}$$

Now note,

$$\frac{\partial^2 h}{\partial w_1 \partial \overline{w_1}}(0)\,|w_1|^2 = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p)\,(w_1 V^0)_j\,\overline{(w_1 V^0)}_k$$

$$= -\kappa |w_1|^2 \qquad \text{——— } \underline{(2)}$$

for some $\kappa > 0$. Thus, we see that $\exists\ \delta \in (0,\varepsilon)$ sufficiently ~~small that~~ small and a $\lambda_0 > 0$ sufficiently small that

- ∗ $\widetilde{\rho}(w) < 0 \ \forall\ w \in (\overline{D(0;\delta)} \setminus \{0\}) \times \{0\}$
- ∗ $\widetilde{\rho}(w) < 0 \ \forall\ w \in \overline{D(0;\delta)} \times \{(0,\dots,0,-\lambda)\},\ \forall\ \lambda \in (0,\lambda_0)$.

By continuity of $\widetilde{\rho}$, we can find $R > 0$ ~~and~~ a $\lambda \in (0,\lambda_0)$ and an $r > 0$ <u>such that</u>:

$$\widetilde{\rho}(w) < 0 \ \text{ on } H := D(0;\delta) \times \mathbb{B}^{n-1}(q_\lambda;\lambda/2) \cup \mathrm{Ann}(0;\delta - r, \delta + r) \times \mathbb{B}^{n-1}(q_\lambda;R),$$

<u>where</u> $R > \lambda$, and

$$q_\lambda := (0,\dots,0,-\lambda).$$

Note that, by construction $D(0;\delta) \times \mathbb{B}^{n-1}(q_\lambda;R) \not\subset \Phi(\Omega \cap \mathbb{B}^n(0;\varepsilon))$ by one of Hartogs' theorems, for any $f \in \mathcal{O}(\Omega)$,

$$(f \circ \Phi^{-1})\big|_H \ \text{ extends to a holomorphic function on } D(0;\delta) \times \mathbb{B}^{n-1}(q_\lambda;R).$$

This violates the fact that Ω is a domain of holomorphy. Hence, $\underline{(1)} \Rightarrow \underline{(6)}$ is established.