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LECTURE 5

* Subharmonicity cont'd...

A careful look at Theorem 4.4 & Remark 4.5 reveals that we have actually proved the following:

* Theorem 5.1. Let $\Omega \subset \mathbb{C}$ be an open set and let $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be u.s.c. on Ω. Then, the following are equivalent:

1) u is subharmonic on Ω
2) For each $z \in \Omega$ ~~such that~~

$$u(z) \leqslant \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta \quad \forall r>0 \text{ s.t. } \overline{D(z;r)} \subset \Omega.$$

3) For each $z \in \Omega$, $\exists R(z) > 0$ such that

$$u(z) \leqslant \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta \quad \forall r \in (0, R(z)).$$

* The two slightly different forms of the Mean Value Inequality displayed by (2) and (3) are very useful; the STRONGER form (2) is useful in theorems/proofs where subharmonicity of $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *a priori* known. The WEAKER form (3) is used to check whether a given u is subharmonic. For instance:

* Proposition 5.2. Subharmonicity is a local property. I.e., if $\Omega \subset \mathbb{C}$ is open and $u : \Omega \to \mathbb{R} \cup \{-\infty\}$, then

$$u \in sh(\Omega) \Leftrightarrow \text{for each } a \in \Omega,\ \exists \text{ a nbhd. } V_a \ni a \text{ such that } \left(u\big|_{V_a}\right) \in sh(V_a).$$

Proof: Assume $u \in sh(\Omega)$. Then, for any $a \in \Omega$, $V_a := \Omega$ satisfies the second statement.

Now, suppose the second statement holds true. Then $u \in usc(\Omega)$ since upper semi-continuity is a local property. Pick any $z \in \Omega$ and let V_z be as described. Then, by Theorem 5.1./(2), if we set

$$R(z) := \sup\{r > 0 : \overline{D(z;r)} \subset V_z\},$$

Then:

$$u(z) \leqslant \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta \quad \forall r \in (0, R(z)).$$

Hence, by Theorem 5.2./(3), u is subharmonic on Ω. [Proved]

* Before we can demonstrate interesting examples, we will need the following two results, for whose proofs we shall not have time. Both results are closely related, using similar methods of proof. The second result below

is a consequence of the first.

Theorem 5.3. Let ~~u~~ $\Omega \subset \mathbb{C}$ be an open set and let $u \in sh(\Omega)$. For $\varepsilon > 0$, define

$$\Omega_\varepsilon := \{z \in \Omega : \operatorname{dist}[z, \partial\Omega] > \varepsilon\},$$

where $\operatorname{dist}[\cdot, \partial\Omega]$ denotes the Euclidean distance. Then, there exists a family of functions $\{u_\varepsilon : \varepsilon > 0\}$ such that:

* $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon) \cap sh(\Omega_\varepsilon)$, and
* $u_\varepsilon(z) \geqslant u_{\varepsilon^*}(z) \geqslant u(z) \;\forall\, z \in \Omega_\varepsilon$ when $\varepsilon < \varepsilon^*$.

Theorem 5.4. Let $u \in \mathcal{C}^2(\Omega; \mathbb{R})$. Then $u \in sh(\Omega)$ if and only if

$$\Delta u(z) \geqslant 0 \;\forall\, z \in \Omega.$$

* We are now in a position to consider examples of subharmonic functions.

Example 5.5. Let $\Omega \subset \mathbb{C}$ be open. Then, for each $f \in \mathcal{O}(\Omega)$, $|f| \in sh(\Omega)$. The proof follows from Cauchy's Integral formula.

Example 5.6. Let Ω and f be as above. Then $\log|f| \in sh(\Omega)$.
Proof. Define:

$$\mathcal{S} := \{z \in \Omega : f(z) = 0\}.$$

Pick a $z \in \Omega \setminus \mathcal{S}$, which is open in $\mathbb{C}$. Now

$$\Delta(\log|f|)(z) = 4\frac{\partial^2}{\partial z \partial \bar z}(\log|f|)(z) = 2\frac{\partial^2}{\partial z \partial \bar z}[\log(f\bar f)](z)$$

$$= 2\frac{\partial}{\partial z}\left[\frac{\overline{f'}\, f}{|f|^2}\right](z) = 0.$$

Thus $\log|f| \in sh(\Omega \setminus \mathcal{S})$. For any $z \in \Omega \setminus \mathcal{S}$, set

$$R(z) := \sup\{r > 0 : \overline{D(z;r)} \subset \Omega \setminus \mathcal{S}\}.$$

Then, we have [by Theorem 5.1]

$$\left.\begin{aligned} \log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(z + re^{i\theta})|\,d\theta \;\; \forall\, r : 0 < r < R(z) \\ \forall\, z \in \Omega \setminus \mathcal{S}. \end{aligned}\right\} \quad \underline{(5.1)}$$

And clearly

$$\left.\begin{aligned} -\infty = \log|f(z)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \log|f(z + re^{i\theta})|\,d\theta \;\; \forall\, r : \overline{D(z;r)} \subset \Omega, \\ \text{whenever } z \in \mathcal{S}. \end{aligned}\right\} \quad \underline{(5.2)}$$

From (5.1) & (5.2), $\log|f| \in sh(\Omega)$.

⊘ Example 5.7. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex increasing function and suppose $u : \Omega \to \mathbb{R}$ is a bounded subharmonic function. Then $\varphi \circ u \in sh(\Omega)$.

⊘ Exersise 5.8.
PROVE THE ABOVE FACT. See TUTORIAL PROBLEMS, Set 3/#2 for Hints.

* We are now in a position to explore the multivariate analogue of subharmonicity.

* Definition 5.9. Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be an open set and let $u : \Omega \to \mathbb{R} \cup \{-\infty\}$. We say that u is plurisubharmonic on Ω (abbreviated as $u \in psh(\Omega)$) if for each $(a, w) \in \mathbb{C}^n \times \mathbb{C}^n$, defining

$$\omega(a, w) := \{\zeta \in \mathbb{C} : a + \zeta w \in \Omega\},$$

the function $\zeta \longmapsto u(a + \zeta w)$ is subharmonic on those components of $\omega(a, w)$ on which it is NOT identically $-\infty$.

* Remark 5.10. Theorem 5.3 remains true if $\mathbb{C}$ is replaced by $\mathbb{C}^n$, $n \geq 2$, and the word "subharmonic" is replaced by "plurisubharmonic".

* Remark 5.11. In many books, the definition of plurisubharmonicity does not feature the phrase "on those components ... NOT identically $-\infty$. This is because, in several complex variables, subharmonic functions are defined so as to allow them to be identically $-\infty$ on connected components (this is done because we DO WANT to allow plurisubharmonic functions to take the value $-\infty$ along complex lines!), whereas ~~this~~ such functions are excluded in the classical definition. Since we have followed the classical treatment of subharmonicity, we reconcile matters by defining "plurisubharmonicity" as per Definition 5.9.

* Example 5.12. Let Ω be a domain in $\mathbb{C}^n$, $n \geq 2$. Let $f \in \mathcal{O}(\Omega)$ and $f \not\equiv$ const. Then $\log|f| \in psh(\Omega)$.