LECTURE 4 — Date 4/03/2010

Today, we shall survey subharmonic functions. For this, we shall need some definitions:

✱ <u>Definition 4.1.</u> Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R} \cup \{-\infty\}$. We say that u is <u>upper semi-continuous at $a \in \Omega$</u> if

$$\varlimsup_{\substack{z \to a \\ z \neq a}} u(z) \leq u(a).$$

We say that u is <u>upper semi-continuous on Ω</u> (denoted by $u \in usc(\Omega)$) if u is upper semi-continuous at each $a \in \Omega$.

✱ <u>Definition 4.2.</u> Let $\Omega \subset \mathbb{C}$ be an open set and $u \in usc(\Omega)$, $u : \Omega \to \mathbb{R} \cup \{-\infty\}$. Then u is <u>subharmonic</u> on Ω if:

<u>a)</u> $u \not\equiv -\infty$ on any connected component of Ω

<u>b)</u> For any bounded subdomain $\mathscr{D}$ such that $\overline{\mathscr{D}} \subset \Omega$, and any $h \in har(\mathscr{D}) \cap \mathcal{C}(\overline{\mathscr{D}})$ satisfying $u(z) \leq h(z)\ \forall z \in \partial\mathscr{D}$, we have

$$u(z) \leq h(z) \quad \forall z \in \mathscr{D}.$$

We will give an equivalent condition to <u>(a)</u> & <u>(b)</u> above which makes it easier to check if a given $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is subharmonic.

✱ <u>Lemma 4.3.</u> Let $\Omega \subset \mathbb{C}$ be open and $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ subharmonic. Then:

<u>a)</u> The set $\{z \in \Omega : u(z) = -\infty\}$ contains <u>no</u> non-empty open set.

<u>b)</u> For each $(a, r) \in \Omega \times \mathbb{R}_+$ such that $\overline{D(a;r)} \subset \Omega$,

$$\int_0^{2\pi} |u(a + \rho e^{i\theta})|\, d\theta < \infty.$$

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<u>Proof</u>: Suppose <u>(a)</u> is false. Then, we can find a pair $(z_0, r) \in \Omega \times \mathbb{R}_+$ such that

- $\overline{D(z_0; r)} \subset \Omega$,
- $\exists a \in \overline{D(z_0; r)}$ such that $u(a) > -\infty$,
- $u(z) = -\infty\ \forall z \in \mathscr{A} \equiv$ an open sub-arc of $\partial D(z_0; r)$.

We now need the following — Let $u \in usc(\Omega)$.

- <u>Technical fact</u>: There exists a sequence of continuous functions $\{u_\nu\}_{\nu \in \mathbb{N}} \subset \mathcal{C}(\Omega)$ such that $u_\nu(z) \downarrow u(z)$ as $\nu \to \infty$, $\forall z \in \Omega$.

Let $P_{z_0,r}$:= the Poisson kernel associated with $D(z_0;r)$. Then, if we write

$$h_\gamma(z) := \int_0^{2\pi} P_{z_0,r}(z,\theta)\, u_\gamma(z_0+re^{i\theta})\,d\theta, \quad z\in D(z_0;r), \qquad \text{——} \ \underline{(4.1)}$$

then $h_\gamma \in har(D(z_0;r))\bigcap \mathcal{C}(\overline{D(z_0;r)})$, and

$$h_\gamma(z) = u_\gamma(z) \geqslant u(z) \ \forall\, z\in\partial D(z_0;r).$$

By definition, then

$$u(z)\leqslant h_\gamma(z) \ \ \forall\, z \in D(z_0;r). \qquad \text{——} \ \underline{(4.2)}$$

Now, it is easy to check the following:

- Fact: Let $u\in usc(\Omega)$. Then u is bounded above on compact subsets of Ω.

Therefore, ~~wlt~~ by construction:

- $P_{z_0}(z,\cdot)\,u_\gamma(z_0+re^{i\cdot})$ is a decreasing sequence of functions; and
- We may assume that $\exists M>0$ such that
$$u_\gamma(z)\leqslant M \ \ \forall\, z\in\partial D(z_0;r) \ \& \ \forall\, \gamma\in\mathbb{N}.$$

We can therefore apply Fatou's lemma which, in conjunction with (4.1) and (4.2), gives:

$$-\infty < u(a) \leqslant \lim_{\gamma\to\infty} h_\gamma(a) \leqslant \int_0^{2\pi} P_{z_0,r}(a,\theta)\Big[\lim_{\gamma\to\infty} h_\gamma(z_0+re^{i\theta})\Big]d\theta$$
$$= \int_0^{2\pi} P_{z_0,r}(a,\theta)\,u(z_0+re^{i\theta})\,d\theta. \qquad \text{——} \ \underline{(4.3)}$$

But this inequality is impossible if $u(z)=-\infty \ \forall\, z\in$ ~~of~~ an open arc in $\partial D(z_0;r)$. From this contradiction, we realise that (a) must be true.

Part (b) now follows easily. Since the demonstration of (b) ~~IS VERY SIMILAR~~ INVOLVES TECHNIQUES VERY SIMILAR TO THOSE IN THE NEXT THEOREM, WE LEAVE THE PROOF TO THE READER TO COMPLETE.

[Proved]

* Theorem 4.4. Let $\Omega\subset\mathbb{C}$ be open and let $u\in usc(\Omega)$.

u is subharmonic on Ω

$$\Leftrightarrow \left.\begin{array}{l}\text{for each } z\in\Omega,\ \exists R(z)>0 \text{ such that } \overline{D(z;R(z))}\subset\Omega \text{ and} \\ u(z)\leqslant \displaystyle\int_0^{2\pi} u(z+re^{i\theta})\,\frac{d\theta}{2\pi} \quad \forall\, r\in(0,R(z))\end{array}\right\} (*)$$

Proof

The proof of the claim

$$u \text{ is subharmonic} \Rightarrow (*)$$

is already embedded in the proof of Lemma 4.3-(a). I.e. if we fix a $z \in \Omega$ and an $r > 0$ such that $\overline{D(z;r)} \subset \Omega$, we

- Consider a decreasing seq. $\{u_\nu\}_{\nu\in\mathbb{N}} \subset \mathcal{C}(\Omega)$ such that $u_\nu(z) \downarrow u(z)$ as $\nu \to \infty$, $\forall z \in \Omega$; and
- Select $\{u_\nu\}_{\nu\in\mathbb{N}}$ in such a way that $\exists\, M > 0$ such that
$$u_\nu(z) \leq M \quad \forall z \in \partial D(z;r) \ \& \ \forall \nu \in \mathbb{N}.$$

Then, the same argument that leads to inequality (4.3) gives us

$$u(\zeta) \leq \int_0^{2\pi} P_{z,r}(\zeta,\theta)\, u(z+re^{i\theta})\, d\theta \quad \forall \zeta \in D(z;r).$$

In particular, as the above holds for any $r \in \mathbb{R}_+$ such that $\overline{D(z;r)} \subset \Omega$,

$$u(z) \leq \int_0^{2\pi} u(z+re^{i\theta})\, \frac{d\theta}{2\pi} \quad \forall r : 0 < r < R(z) := \sup\{r > 0 : \overline{D(0;r)} \not\subset \Omega\}.$$

Conversely, suppose $(*)$ holds. Then let $\mathcal{D}$ be a subdomain with $\overline{\mathcal{D}} \subset \Omega$. Let $h \in har(\mathcal{D}) \cap \mathcal{C}(\overline{\mathcal{D}})$ such that $h(z) \geq u(z)$ $\forall z \in \partial\mathcal{D}$. We need to show that $h(z) \geq u(z)$ $\forall z \in \mathcal{D}$. Assume this is false, i.e.

$$\exists\, \zeta_0 \in \mathcal{D} \text{ such that } (u-h)(\zeta_0) > 0. \qquad \text{——— (4.4)}$$

~~Let~~ By the fact that $(u-h)$ is upper semicontinuous,

$$\exists\, \zeta_* \overline{\mathcal{D}} \text{ such that } (u-h)(\zeta_*) = \sup_{z\in\overline{\mathcal{D}}} (u-h)(z).$$

By (4.4), $\zeta_* \in \mathcal{D}$. Let $E := \{z \in \mathcal{D} : (u-h)(z) = (u-h)(\zeta_*)\}$.

$E \neq \phi$

E is closed in $\mathcal{D}$ (due to upper semi-continuity).

Pick $a \in E$. Let $R > 0$ be such that

- $\overline{D(a;r)} \subset \mathcal{D}$ $\forall r \in (0,R)$; and
- $R \leq R(a)$.

Suppose $\exists\, \theta_0 \in [0,2\pi]$ such that $(u-h)(a+\rho e^{i\theta_0}) \leq' (u-h)(\zeta_*) - \varepsilon$ for some $\rho \in (0,R)$. Then, there is an interval $I_\varepsilon(\rho) \subset [0,2\pi]$ such that

$$(u-h)(a+\rho e^{i\theta}) \leq (u-h)(\zeta_*) - \varepsilon \quad \forall \theta \in I_\varepsilon(\rho).$$

Then:

$$(u-h)(a) \leq \frac{1}{2\pi}\int_0^{2\pi}(u-h)(a+\rho e^{i\theta})\,d\theta$$
$$\leq \frac{1}{2\pi}\Big[(u-h)(\zeta_*)\big(2\pi-|I_\varepsilon(\rho)|\big)+\big[(u-h)(\zeta_*)-\varepsilon\big]\,|I_\varepsilon(\rho)|\Big]$$
absurd $< (u-h)(\zeta_*) = (u-h)(a)$.

This conclusion holds for any arbitrary $\rho > 0$ subject to the condition $\rho < R$.

I.e. $(u-h)(a+\rho e^{i\theta}) \not< (u-h)(\zeta_*) \quad \forall (\theta, \rho) \in [0, 2\pi] \times (0, R)$.

Therefore $D(a;R) \subset E$, which gives us the fact that E is open.

Since $\mathcal{D}$ is a sub-domain, i.e. $\mathcal{D}$ is connected, we conclude that $E = \mathcal{D}$. Thus

$$\forall z \in \mathcal{D}, \quad (u-h)(z) = (u-h)(\zeta_*) > (u-h)(\zeta) \quad \forall \zeta \in \partial\mathcal{D}.$$

But this contradicts the fact that u is upper semi-continuous at each $\zeta \in \partial\mathcal{D}$.

Hence, $u(z) \leq h(z) \ \forall z \in \mathcal{D}$. This proves that u is subharmonic.

[Proved]

* Remark 4.5. Glancing at the proof of Theorem 4.4, we see that — from the first part of the proof — we can deduce:

THE MEAN VALUE INEQUALITY:

Let Ω be open and $\cancel{u \in S}$ u be subharmonic on Ω. Then, for $z \in \Omega$

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\,d\theta \quad \forall r > 0 \text{ such that } \overline{D}(z;r) \subset \Omega$$