* To study the problem outlined in Lecture 2, we need two more concepts. Firstly

Definition 3.1. A connected open set $\Omega \subset \mathbb{C}^n$ is called a domain of holomorphy if there DOES NOT exist a pair of open sets (Ω_1, Ω_2) with the following properties

(i) $\phi \neq \Omega_1 \subset \Omega_2 \cap \Omega$,

(ii) Ω_2 is connected and $\Omega \not\supseteq \Omega \cup \Omega_2$,

(iii) For each $f \in \mathcal{O}(\Omega)$, $\exists\, F \in \mathcal{O}(\Omega_2)$ such that $F|_{\Omega_1} = f|_{\Omega_1}$.

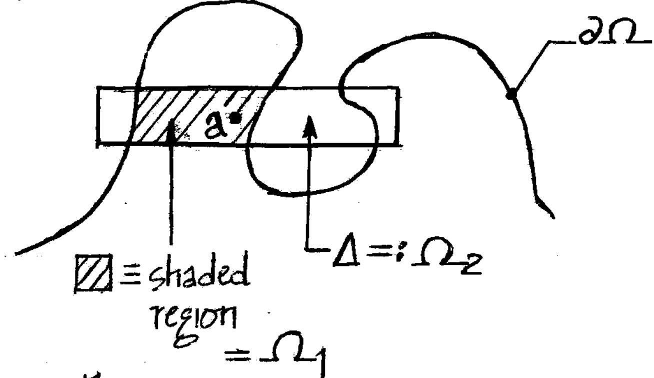
Remark 3.2. Somewhat reminiscent of the 1-variable phenomenon of "analytic continuation along a chain of discs", it can so happen that for some $a \in \Omega$, we can find a polydisc Δ with centre a such that

* for each $f \in \mathcal{O}(\Omega)$, the power series
$$\sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha}(a)(z-a)^\alpha$$
converges in the desired manner on Δ; and
* $\Delta \not\subset \Omega$.

In the drawing on the right, set

$\Omega_1 :=$ the connected component of $\Delta \cap \Omega$ containing a;

$\Omega_2 := \Delta$



Loosely speaking, a domain of holomorphy is a domain $\Omega \subset \mathbb{C}^n$ for which this picture CANNOT be obtained.

Another notion that we shall need — which is of independent interest — is the notion of:

* Cauchy Estimates: Suppose $f \in \mathcal{O}[\Delta(a;\vec{r})]$, and suppose $|f(z)| \leq M$ $\forall z \in \Delta(a;\vec{r})$. Then
$$\left|\frac{\partial^\alpha f}{\partial z^\alpha}(a)\right| \leq \frac{\alpha!\, M}{r^\alpha}.$$

* Exercise 3.3.
PROVE THE CAUCHY ESTIMATES. See TUTORIAL PROBLEMS, Set 2/#1

$\ast$ Proposition 3.4. Let $\Omega \subsetneq \mathbb{C}^n$ be a domain and let

$$\Delta := D(0;R_1)\times \dots \times D(0;R_n)$$

be some fixed, bounded polydisc. Let $K \subset \Omega$ be a compact subset, $\zeta_0 \in \widehat{K}_\Omega$, and let

$$\Omega_2 := \zeta_0 + d_\Delta(K,\Omega^c)\Delta$$

$$\Omega_1 := \text{the connected component of } \Omega_2 \cap \Omega \text{ containing } \zeta_0.$$

For each $f \in \mathcal{O}(\Omega)$, $\exists\, F \in \mathcal{O}(\Omega_2)$ such that $F|_{\Omega_1} = f|_{\Omega_1}$.

Proof: By definition, for each $t \in (0,1)$,

$$x + t d_\Delta(K,\Omega^c)\Delta \subset \Omega \;\forall x \in K$$

Thus, for $x \in K$, by Cauchy estimates

$$\frac{1}{\alpha!}\left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(x)\right| \leq \frac{M_t}{t^{|\alpha|}\left[d_\Delta(K,\Omega)\vec{R}\,\right]^\alpha}$$

where $M_t := \sup\left\{|f(z)| : z \in \bigcup_{x\in K}\left(x + t d_\Delta(K,\Omega^c)\Delta\right)\right\}$. Then

$$\frac{1}{\alpha!}\left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(\zeta_0)\right| \leq \frac{M_t}{t^{|\alpha|}\left[d_\Delta(K,\Omega)\vec{R}\,\right]^\alpha} \quad (\text{by definition of } \widehat{K}_\Omega) \qquad \text{(3.1)}$$

Now, define the power series

$$\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial z^\alpha}(\zeta_0)(z-\zeta_0)^\alpha, \quad z \in \Omega_2. \qquad \text{—— } (\ast)$$

Let Λ be any compact subset of Ω_2. Then, $\exists\, \delta_\Lambda \in (0,1)$ such that

$$\Lambda \subset \zeta_0 + \delta_\Lambda d_\Delta(K,\Omega^c)\Delta.$$

Now, pick and fix a number $t \in (\delta_\Lambda, 1)$. By (3.1)

$$\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\left|\frac{\partial^\alpha f}{\partial z^\alpha}(\zeta_0)\right| |z-\zeta_0|^\alpha \leq \sum_{\alpha\in\mathbb{N}^n}\frac{M_t}{t^{|\alpha|}\left[d_\Delta(K,\Omega^c)\vec{R}\,\right]^\alpha}\left[\delta_\Lambda d_\Delta(K,\Omega^c)\vec{R}\,\right]$$

$$= M_t \sum_{\alpha\in\mathbb{N}^n}\left(\frac{\delta_\Lambda}{t}\right)^\alpha = \frac{M_t}{(1-(\delta_\Lambda/t))^n} \quad \forall z \in \Lambda.$$

This proves in one stroke that $(\ast)$ converges absolutely at each $z \in \Omega_2$ and uniformly on any $\Lambda \subset\subset \Omega_2$.

Thus, we may define

$$F(z) := \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha}(\zeta_0)(z-\zeta_0)^\alpha, \quad z \in \Omega_2,$$

and $F \in \mathcal{O}(\Omega_2)$. Finally, let $\mathcal{D}$ be a polydisc centered at ζ_0 such that $\mathcal{D} \subset \Omega$. By our power-series-development theorem,

$$F|_{\mathcal{D}} = f|_{\mathcal{D}}.$$

Since $\mathcal{D} \subset \Omega_1$, and Ω_1 is connected, $F|_{\Omega_1} = f|_{\Omega_1}$ (by Identity Theorem).

[Proved]

This proposition is the key to the following theorem that solves the problem stated in lecture 2.

* Theorem 3.5. Let $\Omega \subsetneq \mathbb{C}^n$. The following are equivalent:

(1) Ω is a domain of holomorphy

(2) For any bounded, open polydisc Δ centered at 0,
$$d_\Delta(K, \Omega^c) = d_\Delta(\widehat{K}_\Omega, \Omega^c)$$
$\forall K \subset\subset \Omega$.

(3) $\widehat{K}_\Omega$ is compact $\forall K \subset\subset \Omega$.

(4) $\exists f \in \mathcal{O}(\Omega)$ such that it is impossible to find a pair (Ω_1, Ω_2) of open sets with the properties (i) ~ (iii) $\not\!\!\!\!$ in Definition 3.1.

Proof:

(1) $\Rightarrow$ (2) : Suppose (2) is false. Then there exist:

- a polydisc $\Delta \subset\subset \mathbb{C}^n$ with centre at 0; and
- a $K \subset \Omega$, compact, such that
$$d_\Delta(\widehat{K}_\Omega, \Omega^c) < d_\Delta(K, \Omega^c).$$

Then, $\exists\, \zeta_0 \in \widehat{K}_\Omega \setminus K$ such that $d^\Delta_\Omega(\zeta_0) < d_\Delta(K, \Omega^c)$. Thus
$$\Omega_2 := \zeta_0 + d_\Delta(K, \Omega^c)\Delta \not\subset \Omega.$$

If we write:

$\Omega_1 :=$ The connected component of $\Omega \cap \Omega_2$ containing ζ_0,

then by Proposition 3.4, (Ω_1, Ω_2) satisfies (i) ~ (iii) in Definition 3.1. This contradicts (1). Hence (2) must be true.

(2) $\Rightarrow$ (3) is immediately clear.
(3) $\Rightarrow$ (4): Let $\mathcal{D}$ be a countable dense set in Ω; enumerate $\mathcal{D} = \{\xi_j : j \in \mathbb{N}\}$.
Let:

$$\{w_j\}_{j\in\mathbb{N}} := \{\xi_1, \xi_1, \xi_2, \xi_1, \xi_2, \xi_3, \ldots\}.$$

Fix an open polydisc $\Delta \subset\subset \mathbb{C}^n$ with centre 0, and define

$$\Delta_j := d_\Delta(\{w_j\}, \Omega^c).$$

Finally, let $\{K_\nu\}_{\nu\in\mathbb{N}}$ be an exhaustion of Ω by compact subsets of the domain Ω.
By hypothesis, for each $\nu \in \mathbb{N}$, $\exists\, N(\nu)$ such that

$$(\widehat{K_\nu})_\Omega \subset K_\mu \quad \forall \mu \geq N(\nu).$$

By definition, Δ_j is not contained in any K_ν, $\nu \in \mathbb{N}$, and this is true $\forall j \in \mathbb{N}$.
Thus, $\exists\, z_j \in \Delta_j \setminus (\widehat{K_j})_\Omega$ and a function $f_j \in \mathcal{O}(\Omega)$ such that

$$\left.\begin{aligned} \sup_{K_j}|f_j| &< \frac{1}{\;} \\ f_j(z_j) &= 1 \end{aligned}\right\} \qquad (3.2)$$

The above follows from:

Exercise 3.6.
Let $K \subset \Omega$ be compact and $z_0 \notin \widehat{K}_\Omega$. Show that, given any $\varepsilon > 0$, there exists a $f_\varepsilon \in \mathcal{O}(\Omega)$ such that

$$\sup_K |f_\varepsilon| < \varepsilon,$$

$$f_\varepsilon(z_0) = 1.$$

See TUTORIAL PROBLEMS, set 2/#2.

Finally, define

$$f := \prod_{j=1}^{\infty} (1 - f_j)^j. \qquad \text{—— } (3.3)$$

We appeal
to the theory of infinite products to claim that the right-hand side of (3.3) converges in the sense of infinite products. In particular, $f \in \mathcal{O}(\Omega)$ and $f \not\equiv 0$. To see the latter, let $M \in \mathbb{Z}_+$ be so large that

$$(1 - 2^{-j})^j \geq (1 - 4 \times 2^{-j}) > 0 \quad \forall j \geq M.$$

Then
$$|f(z)| \geq \left|\prod_{j=1}^{M}(1-f_j(z))^j\right| \prod_{j\geq M+1}(1-4\times 2^{-j}) \quad \forall z \in K_1,$$
and the infinite product, by classical criteria, is non-zero. Hence $f \not\equiv 0$.

Now, assume there is a pair (Ω_1, Ω_2) with properties (i) ~ (iii) in Definition 3.1. As $\mathcal{D}$ is dense, $\exists\, \Delta_{j0}$ such that $\Delta_{j0} \cap \Omega_1 \neq \phi$. Let $F \in \mathcal{O}(\Omega_2)$ such that $f|_{\Omega_1} = F|_{\Omega_1}$. By construction, we can find a sequence
$$n_1 < n_2 < n_3 < \dots$$
such that

- $z_{n_j} \in \Delta_{j0} \setminus (\widehat{K_{n_j}})_\Omega$
- F vanishes to higher and higher orders at z_{n_j} as $j \uparrow +\infty$.

Then, at any $w_0 \in \overline{\{z_{n_j} : j \in \mathbb{Z}_+\}} \cap \partial\Omega$,
$$\frac{\partial^\alpha F}{\partial z^\alpha}(w_0) = 0 \quad \forall \alpha \in \mathbb{N}^n.$$
But then, $F \equiv 0$, which contradicts the fact that
$$0 \not\equiv f|_{\Omega_1} = F|_{\Omega_1}.$$
Thus, the pair (Ω_1, Ω_2) cannot exist, which establishes (4).

Finally
(4) $\Rightarrow$ (1), by definition.

[Proved]

Date