* Strange analytic-continuation phenomena in $\mathbb{C}^n$, $n \geqslant 2$, cont'd.

Definition 2.1. A domain $\Omega \subset \mathbb{C}^n$ is called a Reinhardt domain if, whenever $z \in \Omega$, $(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n) \in \Omega \ \forall (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$.
(Examples: Any polydisc $\Delta \subset \mathbb{C}^n$ with centre at $0 \in \mathbb{C}^n$; The domain $\Omega \subset \mathbb{C}^n$ considered in "Phenomenon 1".)

The important of Reinhardt domains is that every holomorphic function holomorphic on such a domain is a Laurent series. To be precise:

Theorem 2.2. Let Ω be a Reinhardt domain in $\mathbb{C}^n$. Then, any $f \in \mathcal{O}(\Omega)$ has a Laurent series expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} C_\alpha z^\alpha, \quad z \in \Omega,$$

which converges absolutely at each $z \in \Omega$ and uniformly on compacts of Ω. The coefficients C_α are computed as follows:

$$C_\alpha = \frac{1}{(2\pi i)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(w_1 e^{i\theta_1}, \dots, w_n e^{i\theta_n})\, e^{-i\sum_{j=1}^n \alpha_j \theta_j} \frac{d\theta_1 \dots d\theta_n}{w^\alpha},$$

where w is any point in Ω such that $w_j \neq 0 \ \forall j \leq n$.

Phenomenon 2. Let $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, be a Reinhardt domain with the property that for each $j = 1, \dots, n$, $\exists\, w^{(j)} \in \Omega$ such that $w^{(j)}_j = 0$. Let

$$\widetilde{\Omega} := \{(\rho_1 z_1, \dots, \rho_n z_n) \in \mathbb{C}^n : \rho_j \in [0,1],\ z \in \Omega\}.$$

Then,
for each $f \in \mathcal{O}(\Omega)$, $\exists\, F \in \mathcal{O}(\widetilde{\Omega})$ such that $F|_\Omega = f$.

Proof: Let $f \in \mathcal{O}(\Omega)$. Then, the is a Laurent expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} C_\alpha z^\alpha, \quad z \in \Omega,$$

where the right-hand side converges as stated in Theorem 2.2.
CLAIM. $C_\alpha = 0 \ \forall \alpha \in \mathbb{Z}^n$ s.t. $\alpha_k < 0$ for some $k \leq n$.
To establish this, we assume it is false and reach a contradiction. Hence assume $\exists\, \alpha^0 \in \mathbb{Z}^n$ and some $k \leq n$ such that $C_{\alpha^0} \neq 0$ and $\alpha^0_k < 0$.

By hypothesis, $\exists\, w \in \Omega$ such that $w_k = 0$. Let $\varepsilon > 0$ be so small that

$$\mathbb{B}^n(w;\varepsilon) \subset \Omega.$$

We can find a $\widetilde{w} \in \mathbb{B}^n(w;\varepsilon)$ such that

$$\widetilde{w}_k = 0, \quad \widetilde{w}_j \neq 0 \ \forall j \neq k.$$

Now let $\eta \in \mathbb{N} \to \mathbb{Z}^n$ be some enumeration. By Theorem 2.2,

$$f(z) = \lim_{\nu \to \infty} \sum_{j=0}^{\nu} c_{\eta(j)} z^{\eta(j)} \quad \forall\, z \in \Omega \text{ (absolutely)}$$

In particular: $$\sum_{j=0}^{\infty} \left| c_{\eta(j)} \widetilde{w}^{\eta(j)} \right| < +\infty \qquad \text{—— } \underline{(2.1)}$$

Let N such that $\eta(N) = \alpha^0$. Then:

$$\sum_{j=0}^{\nu} \left| c_{\eta(j)} \widetilde{w}^{\eta(j)} \right| = +\infty \quad \forall\, \nu \geqslant N,$$

which contradicts $\underline{(2.1)}$. Hence our assumption about α^0 must be false, whence the claim.

Now define

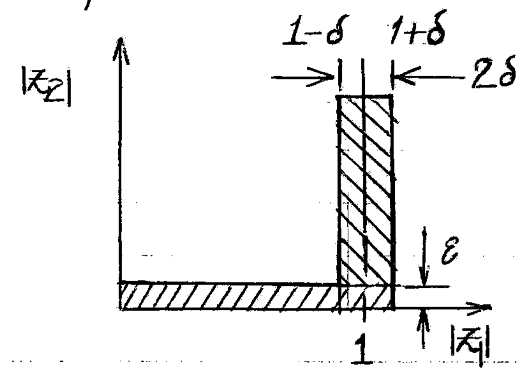
$$F(z) := \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha, \quad z \in \widetilde{\Omega}. \qquad \text{—— } \underline{(2.2)}$$

Since $z = (\rho_1 w_1, \dots, \rho_n w_n)$ for $\underline{\text{some}}$ $w \in \Omega$ and $(\rho_1, \dots, \rho_n) \in [0,1]^n$. Hence, the series on the right-hand side of $\underline{(2.2)}$ converges absolutely for each $z \in \widetilde{\Omega}$. Finally, in view of the Claim above:

$$F\big|_{\Omega} = f.$$

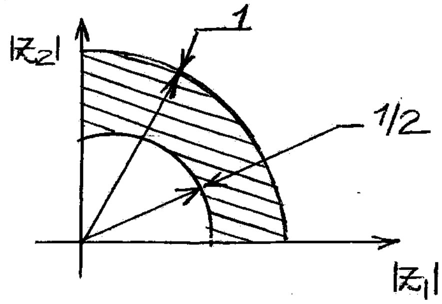
[Proved]

One advantage of working with Reinhardt domains is that they allow pictorial depictions of domains $\Omega \subset \mathbb{C}^2, \mathbb{C}^3$. For example:



$\Omega :=$ The domain discussed in Phenomenon 1

The following picture motivates the next corollary:



$\Omega = \mathbb{B}^2(0;1) \setminus \mathbb{B}^2(0;1/2)$

◎ <u>Corollary to Phenomenon 2</u>

Let $\mathbb{B}^n(0;R)$ denote the <u>Euclidean</u> ball in $\mathbb{C}^n$ with centre 0 and radius R. Let $\Omega := \mathbb{B}^n(0;R) \setminus \overline{\mathbb{B}^n(0;r)}$, $0<r<R$. For each $f \in \mathcal{O}(\Omega)$, $\exists\, F \in \mathcal{O}(\mathbb{B}^n(0;R))$ such that $F|_\Omega = f$.

One of the first things one attempts to study in complex analysis, given any general domain $\Omega \subset \mathbb{C}$, is the properties of the class $\mathcal{O}(\Omega)$ taken collectively. By when $\Omega \subset_{open} \mathbb{C}^n$, $n \geq 2$, the above phenomena show that given certain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, Ω is <u>not</u> the true definition of analyticity of <u>any</u> $f \in \mathcal{O}(\Omega)$. This motivates the following:

<u>PROBLEM</u>: Find a necessary and sufficient condition for a domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, to have the property that ~~for each~~ there exists at least one $f \in \mathcal{O}(\Omega)$ such that

(✱) For <u>any</u> connected open set Ω^* with $\Omega \subsetneq \Omega^*$, f does <u>not</u> admit an $F \in \mathcal{O}(\Omega^*)$ satisfying $F|_\Omega = f$.

✱ <u>Preliminaries to solving the above problem</u>

We begin with a few definitions

◎ <u>Definition 2.3.</u> Let Ω be a domain in $\mathbb{C}^n$, and let $K \subset\subset \Omega$ be a compact subset. The <u>$\mathcal{O}(\Omega)$-hull of K</u> is defined as (and denoted by $\widehat{K}_\Omega$) as follows:

$$\widehat{K}_\Omega := \{ z \in \Omega : |f(z)| \leq \sup_K |f| \;\; \forall f \in \mathcal{O}(\Omega) \}.$$

◎ <u>Definition 2.4.</u> Fix a polydisc $\Delta \subset \mathbb{C}^n$, Δ bounded with centre at $0 \in \mathbb{C}^n$, $n \geq 2$.

Let $\Omega \subset \mathbb{C}^n$ be a domain. We define

$$d^{\Delta}_{\Omega}(z) := \sup\{r>0 : z+r\Delta \subset \Omega\}.$$

It is easy to see that: <u>(a)</u> If $\Omega \subsetneq \mathbb{C}^n$, then $d^{\Delta}_{\Omega} < \infty$; and <u>(b)</u> If $\Omega \neq \mathbb{C}^n$, then $z \mapsto d^{\Delta}_{\Omega}(z)$ is continuous on Ω. Hence, for any $K \subset \Omega$ closed, if we define (here $\Omega \subsetneq \mathbb{C}^n$)

$$d_{\Delta}(K, \Omega^c) := \inf_{z\in K} d^{\Delta}_{\Omega}(z).$$

Then:

$$d_{\Delta}(K, \Omega^c) > 0 \quad \text{whenever } K \text{ is compact.}$$