Before we approach the concept of a "holomorphic function" let us establish some notation:

α is a multi-index, i.e. $\alpha \in (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$

$|\alpha| := \sum_{j \leq n} \alpha_j$, $\quad \alpha! := \alpha_1! \alpha_2! \dots \alpha_n!$

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad j = 1, \dots, n$$

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \quad j = 1, \dots, n$$

$$z^\alpha := \prod_{j \leq n} z_j^{\alpha_j}$$

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} := \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$$

* The concept of holomorphicity

We will propose several definitions that are obvious generalizations to $\mathbb{C}^n$, $n \geq 2$, of the classical concept of holomorphicity. So, let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be an open subset. Recall that in 1-variable, initially:

- we require $f \in \mathcal{C}^1(G)$ when we say that f is holomorphic on $G \subset \mathbb{C}$;
- later we see, via Goursat's Theorem, that the requirement that $f \in \mathcal{C}^1(G)$ is redundant.

Similarly, in proposing the next three definitions, IT IS ASSUMED THAT $f \in \mathcal{C}^1(\Omega)$ IN (H1) — (H3) BELOW.

H1) $f : \Omega \to \mathbb{C}$ is said to be holomorphic if
$$\frac{\partial f}{\partial \bar{z}_j}(z) = 0 \quad \forall z \in \Omega, \ j = 1, 2, \dots, n.$$

H2) $f : \Omega \to \mathbb{C}$ is said to be holomorphic if f is holomorphic in each variable separately, i.e. for each $(a, j) \in \Omega \times \{1, \dots, n\}$, setting
$$\omega(a, j) := \{\zeta \in \mathbb{C} : (a_1, \dots, a_{j-1}, \zeta, a_{j+1}, \dots, a_n) \in \Omega\},$$
$f_{a,j} : \zeta \mapsto f(a_1, \dots, a_{j-1}, \zeta, a_{j+1}, \dots, a_n)$ is holomorphic on $\omega(a, j)$.

H3) $f : \Omega \to \mathbb{C}$ is said to be holomorphic if for each $a \in \Omega$ and each polydisc $\Delta(a; \vec{r}) := D(a_1; r_1) \times \dots \times D(a_n; r_n) \subset \Omega$, there is a power-series development

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z-a)^\alpha \ , \quad z \in \Delta(a;\vec{r})$$

where the series on the right-hand side converges absolutely at each $z \in \Delta(a;\vec{r})$, and converges uniformly on each compact subset $K \subset \Delta(a;\vec{r})$.

Observe how each definition reduces to the same notion of holomorphicity when we set $n=1$! Are (H1) – (H3) equivalent when $n \geq 2$? The answer is provided by

* Theorem 1.1. Let Ω be an open subset of $\mathbb{C}^n$, $n \geq 2$, and let $f: \Omega \to \mathbb{C}$ be of class $\mathcal{C}^1$. The following are equivalent:

1) $\dfrac{\partial f}{\partial \bar{z}_j}(z) = 0 \ \forall z \in \Omega$, $j = 1, \dots, n$.

2) f is holomorphic in each variable separately.

3) For each $a \in \Omega$ and each polydisc $\Delta(a;\vec{r}) \subset \Omega$, there is a power-series development

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z-a)^\alpha, \quad z \in \Delta(a;\vec{r}),$$

where the series converges absolutely at each $z \in \Delta(a;\vec{r})$ and uniformly on each compact subset $K \subset \Delta(a;\vec{r})$.

Sketch of proof:

1) $\Rightarrow$ 2) : This is quite simple; let $(a,j) \in \Omega \times \{1,2,\dots,n\}$ and recall that holomorphicity of $f_{a,j}$ on $\omega(a,j) \Leftarrow \dfrac{\partial f_{a,j}}{\partial \bar{\zeta}}(\zeta) = 0 \ \forall \zeta \in \omega(a,j)$

$$\Leftrightarrow \frac{\partial f}{\partial \bar{z}_j}(a_1, \dots, a_{j-1}, \zeta, a_{j+1}, \dots a_n) = 0$$

$$\forall \zeta \in \omega(a,j).$$

2) $\Rightarrow$ 3) Follows if we can show

Lemma 1.2. Assuming (2), if $\Delta(a;\vec{r})$ is such that $\overline{\Delta(a;\vec{r})} \subset \Omega$, then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D(a_1;r_1)} \cdots \int_{\partial D(a_n;r_n)} \frac{f(\zeta)}{\prod_{j=1}^{n} (\zeta_j - z_j)} \, d\zeta_n \, d\zeta_{n-1} \cdots d\zeta_1 \quad \text{— (1.1)}$$

for every $z \in \Delta(a;\vec{r})$.

<u>Proof</u>: By <u>(2)</u>, we may apply Cauchy's integral formula to get

$$f(z) = f_{a,1}(z_1) = \frac{1}{2\pi i}\int_{\partial D(a_1;r_1)} \frac{f(\zeta_1, z_2, \dots, z_n)}{(\zeta_1 - z_1)}\, d\zeta_1 \qquad \underline{(1.2)}$$

Now define, for any $\vec{\zeta} \in \partial D(a_1;r_1) \times \dots \times \partial D(a_n;r_{k-1})$, $k \geq 2$.

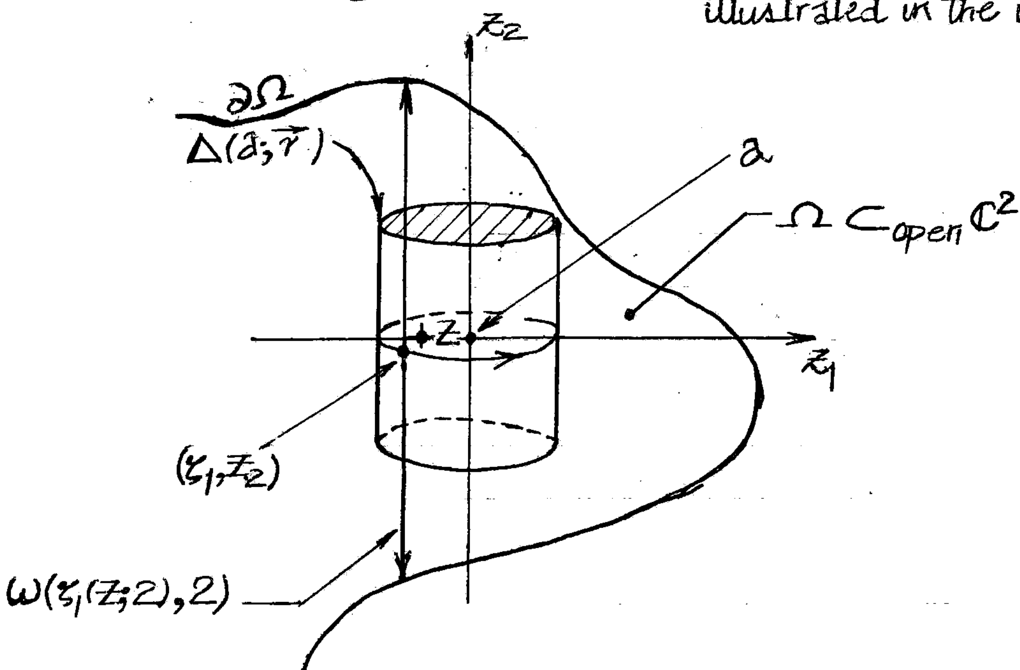
$$\zeta(z;k) := (\zeta_1, \dots, \zeta_{k-1}, z_k, z_{k+1}, \dots, z_n) \in \overline{\Delta(a;\vec{r})} \subset \Omega.$$

Eqn. <u>(1.2)</u> may be rewritten as:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(a_1;r_1)} f_{\zeta_1(z;2),2}(z_2)\Big/(\zeta_1 - z_1)\, d\zeta_1$$

$$= \frac{1}{(2\pi i)^2}\int_{\partial D(a_1;r_1)}\int_{\partial D(a_2;r_2)} \frac{f(\zeta_1,\zeta_2, z_3, \dots, z_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_2\, d\zeta_1 \quad \left\{\begin{array}{l}\text{since, by (2),}\\ f_{\zeta_1(a;2),2} \text{ is holo.}\end{array}\right.$$

Iterating this argument, we prove this lemma [the principle of which is illustrated in the drawing below] ▨



∗ <u>Exercise 1.3.</u>

WE LEAVE IT AS A PROBLEM FOR THE TUTORIAL TO SHOW HOW EQUATION <u>(1.1)</u> $\Rightarrow$ <u>(3)</u>. See <u>TUTORIAL PROBLEMS</u>, set 1/#1.

3) $\Rightarrow$ 1) Suppose $z_0 \in \Omega$. Fix some $\Delta(z_0;\vec{r})$ such that $\Delta(z_0;\vec{r}) \subset \Omega$. By assumption

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha \quad \forall z \in \Delta(z_0;\vec{r}),$$

and the conditions of convergence are such that — in EXACTLY the same manner as was shown for one complex variable — we can differentiate term-by-term. So

$$\frac{\partial f}{\partial \bar{z}_j}(z_0) = \frac{\partial}{\partial \bar{z}_j}\left[\sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha\right]_{z = z_0}$$

$$= \sum_{(\alpha : \alpha_k = 0 \; \forall k \neq j)} \left\{ a_\alpha \frac{\partial (z_j - z_0)^{\alpha_j}}{\partial \bar{z}_j} \right\} = 0$$

Since z_0 was arbitrary, this establishes (3).

[Proved]

* ~~Remark 1.4. Suppose $a \in \Omega$~~

Note that now, any one of the definitions (H1) — (H3) may be taken as the definition of holomorphicity. Now that we have a definition, if Ω is an open set in $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}$ is holomorphic, we will denote this fact as $f \in \mathcal{O}(\Omega)$. So:

$$\mathcal{O}(\Omega) := \{ f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic on } \Omega \}$$

As in the one-variable theory, $\mathcal{O}(\Omega)$ will be endowed with the topology of uniform convergence on compacts.

* Remark 1.4. Suppose Ω is open in $\mathbb{C}^n$, $n \geqslant 2$, $a \in \Omega$, and $f \in \mathcal{O}(\Omega)$. Suppose $\Delta(a;\vec{r})$ is such that $\Delta(a;\vec{r}) \subset \Omega$. In the power-series development

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - a)^\alpha, \quad z \in \Delta(a;\vec{r}),$$

the coefficients a_α depend on a, but not on $\Delta(a;\vec{r})$. Since we can differentiate term-by-term, we get

$$a_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha}(a).$$

* Remark 1.5. All results in one-variable complex analysis whose proofs are unaffected if subscripts are replaced by multi-indices, or if the multi-variable analogues are reducible to one variable, ~~hold~~ extend to $\mathbb{C}^n$, $n \geq 2$. These include:

- Liouville's Theorem
- Open Mapping Theorem
- Maximum Modulus Theorem
- Weierstrass' Theorem on limits of holomorphic functions

* Exercise 1.6.
PROVE THE OPEN MAPPING THEOREM FOR DOMAINS IN $\mathbb{C}^n$, $n \geq 2$.
See TUTORIAL PROBLEMS, set 1/#2.

Certain results have slightly different statements. For example:

* Identity Theorem. Let Ω be a domain in $\mathbb{C}^n$, $n \geq 2$, and let $f \in \mathcal{O}(\Omega)$. If f vanishes on a non-empty open subset of Ω, then f is identically zero on Ω.

* Exercise 1.7
WE WILL LEAVE THE PROOF OF THE IDENTITY THEOREM AS A PROBLEM FOR THE TUTORIAL. See TUTORIAL PROBLEMS, set 1/#3.

* Remark 1.8. In the 1-variable analogue of the Identity Theorem, the statement "f vanishes on a non-empty open subset" is replaced by the MUCH WEAKER hypothesis "f vanishes on a set having a limit point in Ω". The hypothesis must be stronger in n-variables, $n \geq 2$, not just because the proof demands it, but because of the following:

- DEEP RESULT:
  Let Ω be open in $\mathbb{C}^n$, $n \geq 2$, and let $f \in \mathcal{O}(\Omega)$. If $f^{-1}\{0\} \neq \phi$, then $f^{-1}\{0\}$ cannot be discrete.

This deep result follows, as a corollary to a certain highly anomalous analytic-continuation phenomenon. This is our next topic of study.

* Strange analytic-continuation phenomena in $\mathbb{C}^n$, $n \geq 2$

Phenomenon 1: Let $\Omega := \mathbb{D} \times D(0;\varepsilon)^{n-1} \cup \text{Ann}(0;1-\delta,1+\delta) \times \mathbb{D}^{n-1} \subset \mathbb{C}^n$, $n \geq 2$, where $0 < \varepsilon, \delta \ll 1$. For each $f \in \mathcal{O}(\Omega)$, $\exists F \in \mathcal{O}(\mathbb{D}^n)$ such that $F|_{\Omega \cap \mathbb{D}^n} = f|_{\Omega \cap \mathbb{D}^n}$.

Proof: Define the function $F : \mathbb{D}^n \to \mathbb{C}$ by

$$F(z_1,\ldots,z_n) := \int_{\partial\mathbb{D}} \frac{f(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} \frac{d\zeta}{2\pi i} \quad \forall z \in \mathbb{D}^n.$$

Check that the conditions for differentiating under the integral sign are satisfied. Hence

$$\frac{\partial F}{\partial \bar{z}_j}(z_1,\ldots,z_n) = \int_{\partial\mathbb{D}} \frac{1}{\zeta - z_1} \frac{\partial f}{\partial \bar{z}_j}(\zeta, z_2, \ldots, z_n) \frac{d\zeta}{2\pi i}$$

$$= 0 \text{ [because } f \text{ is holomorphic]}, \quad j = 2, \ldots, n.$$

$$\frac{\partial F}{\partial \bar{z}_1}(z_1,\ldots,z_n) = \int_{\partial\mathbb{D}} \frac{\partial}{\partial \bar{z}_1}\left(\frac{1}{\zeta - z_1}\right) f(\zeta, z_2, \ldots, z_n) \frac{d\zeta}{2\pi i}$$

$$= 0 \text{ [because } \zeta - z_1 \neq 0 \ \forall (\zeta, z_1) \in \partial\mathbb{D} \times \mathbb{D}].$$

Hence, $F \in \mathcal{O}(\mathbb{D}^n)$. Now note

$$F(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f_{z,1}(\zeta)}{\zeta - z_1} d\zeta = f_{z,1}(z_1) \quad \text{[Cauchy Integral Formula]}$$

$$= f(z) \quad \forall z \in \mathbb{D} \times D(0;\varepsilon)^{n-1}$$

Thus:

$$\left(F|_{\Omega} - f\right) \text{ vanishes on } \mathbb{D} \times D(0;\varepsilon)^{n-1}.$$

As $\Omega \cap \mathbb{D}^n$ connected, $F|_{\Omega \cap \mathbb{D}^n} = f|_{\Omega \cap \mathbb{D}^n}$.

[Proved]

Remark 1.9. The above is in complete contrast to what happens in one complex variable. In one variable: Let $G \subset \mathbb{C}$ be an open set. For any $\widetilde{G} \supsetneq G$, $\exists f_{\widetilde{G}} \in \mathcal{O}(G)$ but admits no holomorphic extension to $\widetilde{G}$.