UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2022

HOMEWORK 1

Instructor: GAUTAM BHARALI **Assigned: JANUARY 14, 2022**

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1. Consider the following axioms:

- **Singleton sets axiom.** If x is an element, then the collection whose only element is x — which is denoted by $\{x\}$ — is a set.
- **Pair sets axiom.** If x and y are elements, then the collection whose elements are precisely x and y — which is denoted by $\{x, y\}$ — is a set.

Now, let X and Y be non-empty sets.

(a) Using the above axioms and any other axioms of Set Theory presented in class, show that ordered pairs exist in the form of appropriate pair sets.

(b) Using (a) and suitable axioms of Set Theory presented in class, show that $X \times Y$ is a set.

2. Show that for any natural number n, $S(n) \neq n$. (Here, $S(\cdot)$ denotes the successor as postulated by Peano's axioms.)

3. Prove, using Peano's axioms, that if $\Sigma(n)$ denotes some statement involving the natural number n, and if

- $\Sigma(1)$ is true, and
- whenever $\Sigma(n)$ is true, then $\Sigma(S(n))$ is true,

then $\Sigma(n)$ is true for every natural number $n \neq 0$.

4. Let X and Y be two non-empty sets and let $f, g : X \to Y$ be two functions. Why do we define $f = g$ as

$$f(x) = g(x) \ \ \forall x \in X\,?$$

Be sure that you give a reason originating in the fundamentals!

5. Let $a \setminus b$ and $c \setminus d$ be two integers ($a, b, c, d \in \mathbb{N}$). Recall that:

$$(a \setminus b) + (c \setminus d) \ := \ (a + c) \setminus (b + d).$$

Show that this is well-defined: i.e., independent of the choices of a and b representing $a \setminus b$, and of c and d representing $c \setminus d$.

6. The following two problems establish that the operations "+" and "×" defined on $\mathbb{Z}$ extend Peano arithmetic to $\mathbb{Z}$. To this end, **temporarily** denote the addition and multiplication between integers by $+_{\mathbb{Z}}$ and $\times_{\mathbb{Z}}$, respectively.

(a) Define the function $f : \mathbb{N} \to \mathbb{Z}$ by $f(n) := n \setminus 0$ for each $n \in \mathbb{N}$. Show that f is injective.

(b) Show that

$$\begin{aligned} f(m+n) &= f(m) +_{\mathbb{Z}} f(n), \\ f(m \times n) &= f(m) \times_{\mathbb{Z}} f(n), \quad \forall m, n \in \mathbb{N}. \end{aligned}$$

7. This problem shows why the collection

$$\mathfrak{U} := \text{ the collection of all sets,}$$

is **not** a set (or, alternatively, that one **cannot** declare $\mathfrak{U}$ to be a set by an axiom that is compatible with the other axioms of Set Theory).
To this end:

(a) Assume that $\mathfrak{U}$ **is** a set. Then explain why

$$A := \{S \in \mathfrak{U} : S \notin S\}$$

is a set.

(b) Does $A \in A$ or $A \in (\mathfrak{U} \setminus A)$? Based on this, argue why $\mathfrak{U}$ is not a set.

Remark. The outcome of the question in (b) above is called *Russell's Paradox*.