PICK INTERPOLATION ON THE POLYDISC: SMALL FAMILIES OF SUFFICIENT KERNELS

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Abstract. We give a solution to Pick's interpolation problem on the unit polydisc in $\mathbb{C}^n$, $n \geq 2$, by characterizing all interpolation data that admit a $\mathbb{D}$-valued interpolant, in terms of a family of positive-definite kernels parametrized by a class of polynomials. This uses a duality approach that has been associated with Pick interpolation, together with some approximation theory. Furthermore, we use duality methods to understand the set of points on the n-torus at which the boundary values of a given solution to an *extremal* interpolation problem are not unimodular.

1. Introduction, some preliminaries, and a statement of results

The interpolation problem referred to in the title is as follows:

($*$) Let $X_1, \dots, X_N$ be distinct points in the polydisc $\mathbb{D}^n$ and let $w_1, \dots, w_N \in \mathbb{D}$. Find a necessary and sufficient condition on the data $\{(X_j, w_j) : 1 \leq j \leq N\}$ such that there exists a holomorphic function $F : \mathbb{D}^n \longrightarrow \mathbb{D}$ satisfying $F(X_j) = w_j,\ j = 1, \dots, N$.

Here, and elsewhere in this paper, $\mathbb{D}$ denotes the open unit disc with centre $0 \in \mathbb{C}$. We begin by discussing some of the ideas and results that have influenced our theorems below (although our overview of those ideas will be slightly ahistorical). We must begin by stating that the ideas alluded to have a close connection to the work of Cole, Lewis and Wermer [7] (also see [8] by Cole and Wermer) on the existence of interpolants in a given uniform algebra for an interpolation problem between its maximal ideal space and $\mathbb{D}$.

At the heart of the works [7] and [8] is a method, which goes back to Sarason [18], of representing the quotient of a uniform algebra by a closed ideal as an algebra of operators on some Hilbert space. It turns out that a formula for the quotient norm in such a setting — which derives from the representation alluded to — can be transported to the setting of *dual algebras* and their quotients by *weak* closed ideals*. In [15], McCullough provides such a formula. He further uses the insights gained in proving this formula in such a way as to *also* address the existence of interpolants in $H^\infty(\mathbb{D}^n)$ for the problem ($*$).

Let us elaborate upon the phrase "dual algebra". Given a complex, separable Hilbert space H, let $\mathcal{B}(H)$ be the space of bounded operators on H. It is known that the dual of the space of trace class operators of H is isometrically isomorphic to $\mathcal{B}(H)$ (endowed with the operator-norm topology). Via this isomorphism, one can make sense of the weak* topology on $\mathcal{B}(H)$. A unital subalgebra $\mathscr{A}$ of $\mathcal{B}(H)$ is called a *dual algebra* if it is weak* closed. Our interest in dual algebras stems from the fact that $H^\infty(\mathbb{D}^n)$ — the class of all bounded holomorphic functions on $\mathbb{D}^n$ — is a dual algebra. Hence, let us specialize to $\mathbb{D}^n$. Write:

$$A(\mathbb{D}^n) := \mathcal{C}(\overline{\mathbb{D}^n}; \mathbb{C}) \cap \mathcal{O}(\mathbb{D}^n),$$
$$\mathbb{T}^n := (\partial\mathbb{D})^n \subsetneq \partial(\mathbb{D}^n) \quad \text{and} \quad m = \text{the normalized Lebesgue measure on } \mathbb{T}^n. \tag{1.1}$$

Recall that the classical Hardy space $H^2(\mathbb{T}^n)$ is the closure in $\mathbb{L}^2(\mathbb{T}^n, dm)$ of $A(\mathbb{D}^n)|_{\mathbb{T}^n} := \{f|_{\mathbb{T}^n} : f \in A(\mathbb{D}^n)\}$. The space of all multipliers preserving $H^2(\mathbb{T}^n)$ is $H^\infty(\mathbb{D}^n)$. (The

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functions in $H^2(\mathbb{T}^n)$ and $H^\infty(\mathbb{D}^n)$ have different domains of definition, but we assume that readers know how this apparent problem is dealt with — and refer them to Section 3 if they don't.) Viewed as a subalgebra of $\mathcal{B}(H^2(\mathbb{T}^n))$, it is known that $H^\infty(\mathbb{D}^n)$ is a dual algebra. In view of the discussion above, with $H = H^2(\mathbb{T}^n)$, it is meaningful to talk about the weak* closure of a subalgebra of $H^\infty(\mathbb{D}^n)$.

We now have almost all the background needed to present our first theorem, and to introduce a result that has strongly influenced this theorem. We first fix some notation. We will always use $\mathfrak{A}$ to denote a uniform subalgebra of $A(\mathbb{D}^n)$. Given $g \in \mathbb{L}^2(\mathbb{T}^n, dm)$, we shall set

$$\mathfrak{A}^2(g) := \text{the closure of } \mathfrak{A}|_{\mathbb{T}^n} \text{ in } \mathbb{L}^2(\mathbb{T}^n, |g|^2 dm).$$

The following spaces associated to $\mathfrak{A}$ are very useful in the discussion of Pick interpolation in higher dimensions:

$$^{\perp}\mathfrak{A} := \Big\{ f \in \mathbb{L}^1(\mathbb{T}^n) : \textstyle\int_{\mathbb{T}^n} \psi f\, dm = 0 \text{ for each } \psi \in \mathfrak{A} \Big\}, \tag{1.2}$$

$$\mathscr{A}(\mathfrak{A}) := (\text{the closure of } \mathfrak{A}|_{\mathbb{D}^n} \text{ in the topology of local unif. convergence}) \cap H^\infty(\mathbb{D}^n). \tag{1.3}$$

Furthermore, we need a definition. (We shall abbreviate $\mathbb{L}^p(\mathbb{T}^n, dm)$ to $\mathbb{L}^p(\mathbb{T}^n)$, $p = 1, 2, \infty$.)

Definition 1.1. Let $\mathfrak{A}$ be a uniform subalgebra of $A(\mathbb{D}^n)$. We say that $\mathfrak{A}$ *has a tame pre-annihilator* if $(\mathcal{C}(\mathbb{T}^n; \mathbb{C}) \cap {}^{\perp}\mathfrak{A})$ is dense in ${}^{\perp}\mathfrak{A}$ in the $\mathbb{L}^1(\mathbb{T}^n)$-norm.

Theorem 1.3 below is strongly motivated by the following result of McCullough. We shall paraphrase it for the case of the polydisc $\mathbb{D}^n$, since this is the representative case, and the argument for the set-up in [15, Theorem 5.12] follows, after a few adjustments, nearly verbatim the argument in the case of $\mathbb{D}^n$.

Result 1.2 (paraphrasing [15, Theorem 5.12] for the case of $\mathbb{D}^n$, and m as in (1.1))**.** *Let $X_1, \dots, X_N$ be distinct points in $\mathbb{D}^n$, $n \geq 2$, and let $w_1, \dots, w_N \in \mathbb{D}$. Fix a uniform algebra $\mathfrak{A} \subseteq A(\mathbb{D}^n)$ having a tame pre-annihilator. Furthermore assume that*

(*a*) *$\mathfrak{A}$ is approximating in modulus, and*
(*b*) *$\overline{K_{\mathfrak{A}}}(X_j, \cdot) \in \mathfrak{A}|_{\mathbb{T}^n}$ for each $j = 1, 2, \dots, N$,*

where $K_{\mathfrak{A}}(x, \cdot)$, $x \in \mathbb{D}^n$, is the Szegő kernel associated with the Hilbert space $\mathfrak{A}^2(1)$. Then, there exists a function $F \in (\mathbb{L}^\infty(\mathbb{T}^n) \cap \mathfrak{A}^2(1))$ with $\sup_{\mathbb{D}^n} |F| \leq 1$ and such that the Poisson integral $\mathbf{P}[F]$ satisfies $\mathbf{P}[F](X_j) = w_j$, for each $j = 1, \dots, N$, if and only if the matrices

$$\Big[(1 - w_j \overline{w}_k) \big\langle K_{\mathfrak{A},\psi}(X_j, \cdot), K_{\mathfrak{A},\psi}(X_k, \cdot) \big\rangle_{\mathfrak{A}^2(\psi)} \Big]_{j,k=1}^N \geq 0, \tag{1.4}$$

for each $\psi \in \mathfrak{A}$ such that $|\psi| > 0$ on $\mathbb{T}^n$, where $K_{\mathfrak{A},\psi}(x, \cdot)$, $x \in \mathbb{D}^n$, is the Szegő kernel associated with the Hilbert space $\mathfrak{A}^2(\psi)$.

We refer the reader to the beginning of Section 2 for a discussion of the term "Szegő kernel associated to a Hilbert space", and of the notation we follow. A uniform subalgebra $\mathfrak{A} \subseteq A(\mathbb{D}^n)$ is said to be *approximating in modulus* if for each non-negative function $g \in \mathcal{C}(\mathbb{T}^n; \mathbb{C})$ and each $\varepsilon > 0$, there exists a $\psi \in \mathfrak{A}$ such that $\sup_{\mathbb{T}^n} \big| |\psi| - g \big| < \varepsilon$.

In its full generality, [15, Theorem 5.12] is an interpolation theorem of the Cole–Lewis–Wermer type. In its paraphrasing as Result 1.2, it is *very* interesting because it solves the problem (∗), with interpolants belonging to the Schur class. Moreover, it does so by providing us with an easier to understand and smaller family of kernels — i.e., those that feature in (1.4) — necessary and sufficient for the existence of an interpolant than those appearing in [7, 8]. (We shall not elaborate any further: interested readers are referred to [15, Proposition 5.9].) It is not possible, when $n \geq 2$, to replace the family of Pick matrices in (1.4) with a single matricial condition as in Pick's well-known solution to (∗) for $n = 1$. Yet, the contrast between Pick's result and the situation when $n \geq 2$ is a constant stimulus to finding a smaller and/or more explicitly defined family of kernels that are necessary and sufficient for the existence of an interpolant. Indeed, this has been among the motivations of works as recent as [11, 13]. This was also our primary motivation for the following (in this paper, $D(a;r)$ will denote the open disc of radius $r > 0$ with centre $a \in \mathbb{C}$):

Theorem 1.3. *Let $X_1, \dots, X_N$ be distinct points in $\mathbb{D}^n$, $n \geq 2$, and let $w_1, \dots, w_N \in \mathbb{D}$. Let $\mathscr{A}$ be a weak* closed subalgebra of $H^\infty(\mathbb{D}^n)$ such that $\mathscr{A} = \mathscr{A}(\mathfrak{A})$ for some uniform subalgebra $\mathfrak{A} \subseteq A(\mathbb{D}^n)$ having a tame pre-annihilator. Fix an integer $R \geq 1$, and define*

$$\mathfrak{P}(R) := \left\{ p \in \mathbb{C}[z_1, \dots, z_n] : p^{-1}\{0\} \cap \overline{D(0;R)}^n = \varnothing \right\}.$$

There exists a function $F \in \mathscr{A}(\mathfrak{A})$ such that $F : \mathbb{D}^n \longrightarrow \mathbb{D}$ and $F(X_j) = w_j$, for each $j = 1, \dots, N$, if and only if the matrices

$$\Big[(1 - w_j\overline{w}_k)\big\langle K_{\mathfrak{A},p}(X_j, \boldsymbol{\cdot}), K_{\mathfrak{A},p}(X_k, \boldsymbol{\cdot})\big\rangle_{\mathfrak{A}^2(p)}\Big]_{j,k=1}^{N} \geq 0 \quad \text{for each } p \in \mathfrak{P}(R), \tag{1.5}$$

where $K_{\mathfrak{A},p}(x, \boldsymbol{\cdot})$, $x \in \mathbb{D}^n$, is the Szegő kernel associated with the Hilbert space $\mathfrak{A}^2(p)$.

Remark 1.4. The hypothesis on $\mathscr{A}$ above holds true for $\mathscr{A} = H^\infty(\mathbb{D}^n)$. It is well known that $H^\infty(\mathbb{D}^n) = \mathscr{A}(A(\mathbb{D}^n))$ (as per our notation in (1.3)). We refer the reader to the end of Section 5 in [15] for a demonstration that $A(\mathbb{D}^n)$ has a tame pre-annihilator. See the first paragraph of Section 6 for a gist of that discussion. In short, Theorem 1.3 provides new information even for the basic problem $(*)$. Secondly, for both the classical problem $(*)$ and when $\mathfrak{A} \subsetneq A(\mathbb{D}^n)$ we give a much more explicit family of kernels than Result 1.2 that are sufficient for interpolation. Indeed, we see that there are *progressively smaller* families of kernels that are sufficient for interpolation. Lastly, Theorem 1.3 is a result of Cole–Lewis–Wermer type, characterizing the existence of interpolants in a variety of unital weak* closed subalgebras of $H^\infty(\mathbb{D}^n)$.

The last sentence of Remark 1.4 needs some explanation. Theorem 1.3 suggests that $\mathscr{A}(\mathfrak{A})$, as defined in (1.3), is weak* closed. In fact, with *no further conditions* on $\mathfrak{A} \subseteq A(\mathbb{D}^n)$, the weak* closure of $\mathfrak{A}$ is $\mathscr{A}(\mathfrak{A})$ — see Proposition 3.6 below.

Before we introduce our next theorem, we ought to mention that the representation, alluded to above, of the quotient of a uniform algebra by a closed ideal as an algebra of operators on some Hilbert space was first proved for $A(\mathbb{D})$ by Sarason in [18]. His approach to Pick interpolation has been very influential. That approach led to Agler's solution of $(*)$ for $n = 2$: see [1] (see also the articles [5] by Ball–Trent and [2] by Agler–McCarthy). There have been a number of articles, based on largely functional-analytic ideas, in the last two decades that have dwelt on the problem $(*)$: we refer the reader to the works listed in the bibliography of [13]. The latter work, we must mention, addresses — using a result of Bercovici–Westwood [6] — the problem of characterizing the existence of interpolants in an arbitrary unital weak* closed subalgebra of $H^\infty(\mathbb{D}^n)$. Our proof of Theorem 1.3 also relies, to an extent, on some of those ideas (and is influenced by [15]). However, it turns out to be very revealing to replace a rather abstract isometry between $\mathscr{A}(\mathfrak{A})$ and an abstract multiplier algebra used in [15] (and elsewhere) by the Poisson integral. In fact, the Poisson integral is the key part of the *mise en place* for versions of Theorem 1.3 for bounded symmetric domains (which will be the subject of forthcoming work). Another difference: in introducing the family $\mathfrak{P}(R)$ we revisit some hands-on computations involving the uniform algebra $A(\mathbb{D}^n)$.

Our next result is aimed at understanding the functions that interpolate the data $\{(X_j, w_j) : 1 \leq j \leq N\}$ for which the interpolation problem $(*)$ is extremal. We say that the problem $(*)$ — given the data $\{(X_j, w_j) : 1 \leq j \leq N\}$ — is *extremal* if it admits an interpolant F for these data with $\sup_{\mathbb{D}^n} |F| = 1$ but admits no interpolant of sup-norm less than 1.

The specific form of Theorem 1.7 below is motivated, in part, by a result of Amar and Thomas [4] (see below), and by the fact that the generic extremal problem for the bidisc, and with $N = 3$, has a unique solution that is a rational inner function — see [3, Theorem 12.13]. Some interesting results on the extremal problem in higher dimensions, but still with $N = 3$, were obtained recently by Kosiński [14]. Little is currently known when $N \geq 4$. It is not even known whether, for a generic extremal problem, there exists an interpolant that (generalizing the situation in the bidisc) is an inner function. A bounded holomorphic function f on $\mathbb{D}^n$ is called an *inner function* if the values of the radial boundary-value function $f^\bullet$, defined as

$$f^\bullet(\zeta) := \lim_{r\to 1^-} f(r\zeta) \quad (\text{for } m\text{-a.e. } \zeta \in \mathbb{T}^n), \tag{1.6}$$

are unimodular m-a.e. on $\mathbb{T}^n$. We recall here that the fact that the limit on the right-hand side of (1.6) exists m-a.e. on $\mathbb{T}^n$ is the extension of a well-known theorem of Fatou to higher dimensions (see Section 3 for more details).

Amar and Thomas use the phrase "all the points of $\{X_j : 1 \leq j \leq N\}$ are active constraints" to refer to a generic extremal problem on $\mathbb{D}^n$. We shall not define this term here; the reader is referred to [4, Section 0] for a definition. The authors are interested in the nature of the maximum modulus set $M(\phi)$ of an interpolant ϕ for a given extremal problem. To be precise:

Result 1.5 (paraphrasing [4, Theorem 1] for the case of the polydisc)**.** *Let $X_1, \dots, X_N$, distinct points in $\mathbb{D}^n$, $n \geq 2$, and $w_1, \dots, w_N \in \mathbb{D}$ be data for an extremal Pick interpolation problem on $\mathbb{D}^n$. Let ϕ be any interpolant in the Schur class. Write*

$$M(\phi) := \{\zeta \in \mathbb{T}^n : \limsup_{\mathbb{D}^n \ni z \to \zeta} |\phi(z)| = 1\}.$$

Let $[M(\phi)]^\wedge_{A(\mathbb{D}^n)}$ denote the $A(\mathbb{D}^n)$-hull of $M(\phi)$. If all the points of $\{X_j : 1 \leq j \leq N\}$ are active constraints, then $[M(\phi)]^\wedge_{A(\mathbb{D}^n)} \supset \{X_j : 1 \leq j \leq N\}$. In general, $[M(\phi)]^\wedge_{A(\mathbb{D}^n)} \cap \{X_j : 1 \leq j \leq N\} \neq \varnothing$.

The result above describes, in some sense, the size of $M(\phi)$. A natural question that arises from the discussion prior to Result 1.5 is how close the interpolant ϕ is to an inner function. This entails studying the size of the set $\{\zeta \in \mathbb{T}^n : |\phi^\bullet(\zeta)| = 1\}$. Result 1.5 does not quite provide this information and, furthermore, we have the difficulty that

$$M(\phi) \supseteq \{\zeta \in \mathbb{T}^n : |\phi^\bullet(\zeta)| = 1\}.$$

However, some of the tools used in our proof of Theorem 1.3 can be used to obtain information on the set on the right-hand side above. To be more precise, we show that if $\{\zeta \in \mathbb{T}^n : |\phi^\bullet(\zeta)| = 1\}$ is not of full measure, then the set $\mathbb{T}^n \setminus \{\zeta \in \mathbb{T}^n : |\phi^\bullet(\zeta)| = 1\}$ is constrained in a rather specific fashion. Before we can state this theorem, we need the following

Definition 1.6. Let X be a real-analytic manifold. A set $S \subseteq X$ is called a *semi-analytic set* if for each point $p \in S$, there exists an open set $U_p \ni p$ and functions $f_{jk} \in \mathcal{C}^\omega(U_p; \mathbb{R})$, $j = 1, \dots, \mu$, $k = 1, \dots, \nu$, such that

$$S \cap U_p = \bigcup\nolimits_{1 \leq j \leq \mu} \bigcap\nolimits_{1 \leq k \leq \nu} S_{jk},$$

where each S_{jk} is either $\{x \in U_p : f_{jk}(x) = 0\}$ or $\{x \in U_p : f_{jk}(x) > 0\}$.

We are now in a position to state our next theorem.

Theorem 1.7. *Let $X_1, \dots, X_N$ be distinct points in $\mathbb{D}^n$, $n \geq 2$, and let $w_1, \dots, w_N \in \mathbb{D}$. Assume that $(X_1, \dots, X_N; w_1, \dots, w_N)$ are data for an extremal Pick interpolation problem. Let ϕ be any interpolant in the Schur class, and let $\phi^\bullet$ denote the radial boundary-value function of ϕ. Then, the set $\{\zeta \in \mathbb{T}^n : |\phi^\bullet(\zeta)| < 1\}$ is contained in the disjoint union $N \sqcup S$, where N is a set of zero Lebesgue measure and S is the inner limit of a sequence of proper semi-analytic subsets of $\mathbb{T}^n$.*

The proofs of Theorems 1.3 and 1.7 will be presented in Sections 5 and 6, respectively. However, we shall need a few standard facts and a couple of essential propositions before we can give these proofs. Section 2 will be devoted to matters that are primarily functional-analytic in character. Section 4 will be devoted to matters pertaining to function theory in several complex variables.

2. On duality and the weak* topology

This section is intended to present several results, which are primarily functional-analytic in character, that we will need in the proofs of our theorems. Along the way, we shall explain a few terms that had appeared in Section 1 and whose discussion had been deferred.

2.1. **Szegő kernels associated to Hilbert spaces on $\mathbb{T}^n$.** We adopt the notation introduced in Section 1. Let $\mathfrak{A}$ be a uniform subalgebra of $A(\mathbb{D}^n)$, $g \in \mathbb{L}^\infty(\mathbb{T}^n)$ be such that $|g| > c_g$ for some constant $c_g > 0$, and let $\mathfrak{A}^2(g)$ be as defined in Section 1. By construction, $\mathfrak{A}^2(g)$ is a separable Hilbert space with the inner product

$$\langle \psi, \varphi \rangle_g := \int_{\mathbb{T}^n} \psi \bar{\varphi} |g|^2 dm.$$

In this paper, for any $\varphi \in \mathbb{L}^1(\mathbb{T}^n)$, we shall write

$$\mathbf{P}[\varphi] := \text{the Poisson integral of } \varphi.$$

By the properties of g, $\varphi \in \mathbb{L}^1(\mathbb{T}^n)$ whenever $\varphi \in \mathfrak{A}^2(g)$. Thus, for every $x \in \mathbb{D}^n$, we can define $\mathsf{eval}_x : \mathfrak{A}^2(g) \longrightarrow \mathbb{C}$ by

$$\mathsf{eval}_x(\varphi) := \mathbf{P}[\varphi](x).$$

It is routine to show that eval_x is a bounded linear functional for each $x \in \mathbb{D}^n$. Hence, by the Riesz representation theorem, there exists a function in $\mathfrak{A}^2(g)$, which we shall denote in this paper by $\overline{K_{\mathfrak{A},g}}(x, \cdot) : \mathbb{T}^n \longrightarrow \mathbb{C}$, such that

$$\mathsf{eval}_x(\varphi) = \langle \varphi, \overline{K_{\mathfrak{A},g}}(x, \cdot) \rangle_g .$$

We call $K_{\mathfrak{A},g}(x, \cdot)$ the *Szegő kernel associated to* $\mathfrak{A}^2(g)$.

2.2. **General functional analysis.** In this subsection we state a couple of results that are perhaps not widely seen in the form that we need (especially by readers who specialize in complex geometry or function theory). The results themselves are very standard, and we shall only write a line or two about their proofs. For the first such result, we first recall: if X is a Banach space, S is a subspace of X and L is a subspace of X^*, then

$$S^\perp := \{\lambda \in X^* : \lambda(x) = 0 \ \ \forall x \in S\},$$
$$^\perp L := \{x \in X : \lambda(x) = 0 \ \ \forall \lambda \in L\}.$$

Lemma 2.1. *Let X be a Banach space and E, S be closed subspaces of X with $E \subseteq S$. Let $q : S \longrightarrow S/E$ be the quotient map. For each $F \in (S/E)^*$, the map*

$$\Theta : F \longmapsto \widetilde{F \circ q} + S^\perp,$$

where $\widetilde{F \circ q}$ is any (fixed) norm-preserving $\mathbb{C}$-linear extension of $F \circ q$ to X, is well defined and is an isometric isomorphism from $(S/E)^*$ to $E^\perp / S^\perp$.

The proof is utterly standard and runs along the lines of, for instance, [17, Theorem 4.9].

The second result of this subsection is about the dual of the space of trace class operators $\mathscr{T}(H)$, where H and $\mathcal{B}(H)$ are as in Section 1. Our presentation will be very brief, and the reader is referred to [10, Chapter 3, §18] for details of the concepts discussed below.

Given $T \in \mathcal{B}(H)$, write $|T| := (T^*T)^{\frac{1}{2}}$. If we fix an orthonormal basis $\{e_j : j \in \mathbb{N}\}$ of H, the quantity

$$\sum_{j\in\mathbb{N}} \langle |T|e_j,\, e_j\rangle \tag{2.1}$$

is independent of the choice of the orthonormal basis $\{e_j : j \in \mathbb{N}\}$. The space of *trace class operators*, denoted by $\mathscr{T}(H)$, consists of operators $T \in \mathcal{B}(H)$ for which the quantity in (2.1) is finite. Thus, for a fixed $T \in \mathscr{T}(H)$, we have a number

$$\|T\|_{\rm tr} := \sum_{j\in\mathbb{N}} \langle |T|e_j,\, e_j\rangle \tag{2.2}$$

(where $\{e_j : j \in \mathbb{N}\}$ is some orthonormal basis). It is a fact that (2.2) defines a norm and that $\mathscr{T}(H)$ is a Banach space with this norm.

We will need the concept of the trace of an operator in $\mathcal{B}(H)$. One fixes some orthonormal basis on H and attempts a definition as one would for a finite-dimensional H. Convergence and independence of the choice of orthonormal basis hold true for any $T \in \mathscr{T}(H)$. For any such T, we denote the trace by $\mathsf{trace}(T)$. We will not spell out an expression for $\mathsf{trace}(T)$ — we refer the reader to [10, Chapter 3, §18]. What follows from the above procedure is that

$$|\mathsf{trace}(T)| \leq \|T\|_{\rm tr} \quad \forall T \in \mathscr{T}(H). \tag{2.3}$$

It turns out that $\mathscr{T}(H)$ is a two-sided ideal of $\mathcal{B}(H)$. Moreover, given $T \in \mathcal{B}(H)$ and $A \in \mathscr{T}(H)$ we have:

$$\|TA\|_{\rm tr} \leq \|T\|_{\rm op}\|A\|_{\rm tr} \quad \text{and} \quad \|AT\|_{\rm tr} \leq \|T\|_{\rm op}\|A\|_{\rm tr}, \tag{2.4}$$

where $\|T\|_{\rm op}$ represents the operator norm of T. Because of the inequalities above, each $T \in \mathcal{B}(H)$ induces a linear functional $L_T \in (\mathscr{T}(H))^*$ defined by $L_T(A) := \mathsf{trace}(TA)$.

Result 2.2. *The map $\Lambda : \mathcal{B}(H) \longrightarrow (\mathscr{T}(H))^*$ defined by*

$$\Lambda(T) := L_T \quad \forall T \in \mathcal{B}(H),$$

where L_T is defined by $L_T(A) := \mathsf{trace}(TA)$ $\forall A \in \mathscr{T}(H)$, gives an isometric isomorphism of $\mathcal{B}(H)$ onto $(\mathscr{T}(H))^*$.

The above is a standard result; see, for instance, [10, Theorem 19.2].

We end this subsection by reminding ourselves of rank-one operators, which will be needed in the next section. Given $x, y \in H$, we define the rank-one operator $x \otimes y$ as

$$x \otimes y(v) := \langle v, y\rangle x \quad \forall v \in H.$$

It is not hard to see that $x \otimes y \in \mathscr{T}(H)$. Also, we have

$$\|x \otimes y\|_{\rm tr} = \|x\|_H\|y\|_H \quad \text{and} \quad \mathsf{trace}(x \otimes y) = \langle x, y\rangle. \tag{2.5}$$

3. Closure in the weak* topology on $H^\infty(\mathbb{D}^n)$

This section is devoted to providing a simple description of the weak* closure of a uniform subalgebra $\mathfrak{A} \subseteq A(\mathbb{D}^n)$ (and more). Such results are already implicit in the literature — the proofs of the results that we need require known arguments to be assembled properly.

Since, in the discussions that follow, we shall use the term "weak*" in more than one sense, we must make a clarification:

- (•) with H as in Section 2, and for any $\mathbb{C}$-linear subspace $V \subseteq \mathcal{B}(H)$, any mention of the weak* topology or of the properties of V involving the weak* topology, without any further qualification, will refer to the topology that $\mathcal{B}(H)$ acquires as the dual space of $\mathscr{T}(H)$ (which is a consequence of Result 2.2).

Let us now fix our Hilbert space H to be $H^2(\mathbb{T}^n)$. Each $\varphi \in H^\infty(\mathbb{D}^n)$ defines a *multiplier operator* $M_\varphi \in \mathcal{B}(H^2(\mathbb{T}^n))$ as follows. It follows from a result of Marcinkiewicz and Zygmund on multiple Poisson integrals that for any bounded function u on $\mathbb{D}^n$, $n \geq 2$, that is harmonic in each variable separately, the limit

$$\lim_{r\to 1^-} u(r\zeta) =: u^\bullet(\zeta), \;\; \zeta \in \mathbb{T}^n, \text{ exists for } m\text{-a.e. } \zeta \in \mathbb{T}^n; \tag{3.1}$$

see [16, Section 2.3]. When $n = 1$, the latter statement is the classical theorem of Fatou. Furthermore, $u^\bullet$ is of class $\mathbb{L}^\infty(\mathbb{T}^n)$, $n \geq 1$, and satisfies

$$u = \mathsf{P}[u^\bullet] \quad \text{and} \quad \|u^\bullet\|_{\mathbb{L}^\infty(\mathbb{T}^n)} = \sup\nolimits_{\mathbb{D}^n} |u|. \tag{3.2}$$

Since any holomorphic function on $\mathbb{D}^n$ is harmonic in each variable separately, it follows that to each $\varphi \in H^\infty(\mathbb{D}^n)$ is associated the radial boundary-value function $\varphi^\bullet$, which establishes an isometric embedding of $H^\infty(\mathbb{D}^n)$ into $\mathbb{L}^\infty(\mathbb{T}^n)$. With these facts, we have:

$$M_\varphi(h) := \varphi^\bullet h \;\; \forall h \in H^2(\mathbb{T}^n) \quad \text{and} \quad \|M_\varphi\|_{\text{op}} = \sup\nolimits_{\mathbb{D}^n} |\varphi|. \tag{3.3}$$

Thus, the identification

$$H^\infty(\mathbb{D}^n) \ni \varphi \stackrel{\mathsf{j}}{\longmapsto} M_\varphi \in \mathcal{B}(H^2(\mathbb{T}^n)) \tag{3.4}$$

gives an isometric imbedding of $H^\infty(\mathbb{D}^n) \hookrightarrow \mathcal{B}(H^2(\mathbb{T}^n))$. It is known that $H^\infty(\mathbb{D}^n)$ (i.e., identified with $\mathsf{j}(H^\infty(\mathbb{D}^n)) \subset \mathcal{B}(H^2(\mathbb{T}^n))$ as discussed) is weak* closed — e.g., see [15, Lemma 3.6]. Hence, in the remainder of this article, when we discuss properties — or topological operations such as closure — of subsets of $H^\infty(\mathbb{D}^n)$ involving the weak* topology, we shall view them *interchangeably as contained in $H^\infty(\mathbb{D}^n)$ or in $\mathcal{B}(H^2(\mathbb{T}^n))$ without further comment.*

Now, given any $\mathbb{C}$-linear subspace $V \subseteq H^\infty(\mathbb{D}^n)$, the above discussion allows us to define $V^\bullet := \{\varphi^\bullet : \varphi \in V\}$, which is a subspace of $\mathbb{L}^\infty(\mathbb{T}^n)$. Write:

$$\begin{aligned}(V,\ \text{weak*}|_{\mathbb{L}^1}) &:= V^\bullet \text{ relative to the weak* topology on } \mathbb{L}^\infty(\mathbb{T}^n)\\ &\qquad \text{viewed as the dual of } \mathbb{L}^1(\mathbb{T}^n),\\ \text{wk*}(V, \mathbb{L}^1) &:= \text{the closure of } V^\bullet \text{ in the above topology.}\end{aligned}$$

The following two results provide the basis for several useful lemmas that we shall need. Results 3.1 and 3.2 have been stated for $n = 1$ in [3, §3.4]. The proof of Result 3.1 for a general $n \in \mathbb{Z}_+$ can be found in Hamilton's thesis [12, Proposition 4.2.2]. As for Result 3.2: it is standard, and we shall only comment briefly upon its proof.

Result 3.1. *Define* $\mathsf{J} : H^\infty(\mathbb{D}^n)^\bullet \longrightarrow \mathcal{B}(H^2(\mathbb{T}^n))$ *as*

$$\mathsf{J}(\varphi^\bullet) := M_\varphi \quad \forall \varphi \in H^\infty(\mathbb{D}^n).$$

The map J *is a linear isometric embedding of* $H^\infty(\mathbb{D}^n)^\bullet$ *into* $\mathcal{B}(H^2(\mathbb{T}^n))$ *and gives a homeomorphism between* $(H^\infty(\mathbb{D}^n),\ \text{weak*}|_{\mathbb{L}^1})$ *and* $(H^\infty(\mathbb{D}^n), \text{weak*})$.

Result 3.2. *Let* $\{\varphi_\nu\}_{\nu\in\mathbb{N}}$ *be a sequence in* $H^\infty(\mathbb{D}^n)$. *If* $\{\varphi_\nu\}_{\nu\in\mathbb{N}}$ *is weak* convergent, then*

(*i*) $\sup\{\sup_{\mathbb{D}^n} |\varphi_\nu| : \nu \in \mathbb{N}\} < \infty$; *and*
(*ii*) $\{\varphi_\nu(x)\}_{\nu\in\mathbb{N}}$ *converges to* $\varphi(x)$ *for some* $\varphi \in H^\infty(\mathbb{D}^n)$ *and for each* $x \in \mathbb{D}^n$.

The proof of the above relies on the fact that $H^\infty(\mathbb{D}^n)$ is weak* closed. The conclusion (*ii*) follows, essentially, from the latter: just apply $L_{M_{\varphi_\nu}}$ to the rank-one operator $1 \otimes K_{A(\mathbb{D}^n),1}(x,\boldsymbol{\cdot})$, $\nu = 1, 2, 3, \dots$ (where $K_{A(\mathbb{D}^n),1}(x,\boldsymbol{\cdot})$ is as described in subsection 2.1). As for (*i*): it follows from applying the Uniform Boundedness Principle to the collection of linear functionals $\{L_{M_{\varphi_\nu}} : \nu \in \mathbb{N}\} \subset (\mathscr{T}(H^2(\mathbb{T}^n)))^*$ and observing that, by Result 2.2 and (3.3):

$$\|L_{M_{\varphi_\nu}}\|_{\text{op}} = \|M_{\varphi_\nu}\|_{\text{op}} = \sup\nolimits_{\mathbb{D}^n} |\varphi_\nu|.$$

The lemma below now follows almost immediately from Result 3.1.

Lemma 3.3. *Let V be a $\mathbb{C}$-linear subspace of $H^\infty(\mathbb{D}^n)$ and let $\mathcal{V}$ denote the closure of V with respect to the weak* topology. Then* $\mathrm{wk}^*(V, \mathbb{L}^1) = \mathcal{V}^\bullet := \{\varphi^\bullet : \varphi \in \mathcal{V}\}$.

The need for the following result might, at first glance, seem a bit mysterious. Its relevance to the goals of this section is established by Lemma 3.5 below — for whose proof we need Result 3.4.

Result 3.4. *Let X be a separable Banach space and C a convex subset of X^*. Then C is weak* closed if and only if it is sequentially weak* closed.*

The above result is a consequence of the Krein–Šmulian Theorem — see, for instance, [9, Chapter 5, §12] and Corollary 12.7 therein.

Lemma 3.5. *A subspace of $H^\infty(\mathbb{D}^n)$ is weak* closed if and only if it is sequentially weak* closed.*

Proof. Fix a subspace $V \subseteq H^\infty(\mathbb{D}^n)$. Next (in the notation of Result 3.4), set

$$X = \mathbb{L}^1(\mathbb{T}^n) \quad \text{and} \quad C = \mathsf{J}^{-1}(\mathbf{j}(V)),$$

where J and $\mathbf{j}$ are as given by Result 3.1 and (3.4), respectively. By Result 3.1, a sequence $\{\varphi_\nu\} \subset V$ is weak* convergent if and only if $\{\varphi_\nu^\bullet\}$ is convergent in $(V, \text{weak*}|_{\mathbb{L}^1})$. Thus, as $\mathbb{L}^1(\mathbb{T}^n)$ is separable, the lemma follows from Result 3.4 and Lemma 3.3. ☐

The above lemma gives us the main result of this section.

Proposition 3.6. *Let V be a $\mathbb{C}$-linear subspace of $A(\mathbb{D}^n)$. Then:*

(1) *the closure of V in the weak* topology equals*

$$(\textit{the closure of } V|_{\mathbb{D}^n} \textit{ in the topology of pointwise convergence}) \cap \mathbb{L}^\infty(\mathbb{D}^n).$$

(2) *the closure of V in the weak* topology equals*

$$(\textit{the closure of } V|_{\mathbb{D}^n} \textit{ in the topology of local uniform convergence}) \cap \mathbb{L}^\infty(\mathbb{D}^n).$$

In particular, the weak closure of a uniform subalgebra $\mathfrak{A} \subseteq A(\mathbb{D}^n)$ is $\mathscr{A}(\mathfrak{A})$.*

Proof. The proof of (1) is immediate from the last lemma and Result 3.2. Now, given an element φ in the weak* closure of V, any weak* convergent sequence $\{\varphi_\nu\} \subset V$ of which φ is the pointwise limit is — owing to Result 3.2 — uniformly bounded. By the pointwise convergence of the latter sequence and Montel's Theorem, we deduce that $\varphi_\nu \longrightarrow \varphi$ locally uniformly. Hence (2) follows. ☐

4. Some function theory in several complex variables

Although we have used the term "uniform algebra" several times above, it might be helpful to recall the definition. Given a compact Hausdorff space X, a *uniform algebra on X* is a subalgebra of $\mathcal{C}(X; \mathbb{C})$ that is closed with respect to the uniform norm, contains the constants, and separates the points of X. Given a uniform algebra A, we call a subalgebra $B \subset A$ a *uniform subalgebra of A* if B is itself a uniform algebra.

In this paper, we are interested in uniform algebras on $\overline{\mathbb{D}^n}$. We begin with the following result.

Lemma 4.1. *Let $X_1, \dots, X_N$, $N \geq 2$, be distinct points in $\mathbb{D}^n$. Let $\mathfrak{A}$ be a uniform subalgebra of $A(\mathbb{D}^n)$. There exist functions $\Phi_1, \dots, \Phi_N \in \mathfrak{A}$ such that*

$$\Phi_j(X_k) = \boldsymbol{\delta}_{jk}, \;\; j, k = 1, \dots, N,$$

where $\boldsymbol{\delta}_{jk}$ denotes the Kronecker symbol.

The proof of this lemma relies on the fact that $\mathfrak{A}$ separates points on $\mathbb{D}^n$ and is closed under multiplication. We shall skip the proof since it is utterly elementary.

The above lemma is essential to Proposition 4.2, which we shall use several times in Sections 5 and 6. First, we need some notations. Let $X_1, \dots, X_N$ be as in Lemma 4.1 and fix a uniform subalgebra $\mathfrak{A} \subseteq A(\mathbb{D}^n)$. Denote the set $\{X_1, \dots, X_N\}$ by $\mathbf{X}$, and write

$$\mathscr{I}_{\mathfrak{A},\mathbf{X}} := \text{the weak* closure of } I_{\mathfrak{A},\mathbf{X}} \text{ (viz., the ideal of all functions in } \mathfrak{A} \text{ that vanish on } \mathbf{X}).$$

Note that, by Proposition 3.6, each $\psi \in \mathscr{I}_{\mathfrak{A},\mathbf{X}}$ is a bounded holomorphic function. Thus, by the discussion at the beginning of Section 3, the following make sense:

$$\begin{aligned} {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}} &:= \left\{ f \in \mathbb{L}^1(\mathbb{T}^n) : \textstyle\int_{\mathbb{T}^n} \psi^{\bullet} f dm = 0 \text{ for each } \psi \in \mathscr{I}_{\mathfrak{A},\mathbf{X}} \right\}, \\ {}^{\perp}\mathscr{A}(\mathfrak{A}) &:= \left\{ f \in \mathbb{L}^1(\mathbb{T}^n) : \textstyle\int_{\mathbb{T}^n} \psi^{\bullet} f dm = 0 \text{ for each } \psi \in \mathscr{A}(\mathfrak{A}) \right\}. \end{aligned}$$

Hereafter, we shall abbreviate the Szegő kernel associated to $H^2(\mathbb{T}^n)$ — i.e., $K_{A(\mathbb{D}^n),1}(x, \cdot)$ in the notation of subsection 2.1 — to $K(x, \cdot)$, $x \in \mathbb{D}^n$. With this, we state:

Proposition 4.2. *Let $X_1, \dots, X_N$, $N \geq 2$, be distinct points in $\mathbb{D}^n$. Let $\mathfrak{A}$ be a uniform subalgebra of $A(\mathbb{D}^n)$. For each $f \in {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}$, let $[f]$ denote the ${}^{\perp}\mathscr{A}(\mathfrak{A})$-coset of f. There exist constants $a_1, \dots, a_N \in \mathbb{C}$, which are independent of the choice of representative of the coset $[f] \in {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A})$, such that*

$$[f] = \left[\sum_{1 \leq j \leq N} a_j K(X_j, \cdot) \right].$$

Proof. Let us define a linear functional $\mathscr{L}_{[f]} : \mathscr{A}(\mathfrak{A}) \longrightarrow \mathbb{C}$ by

$$\mathscr{L}_{[f]}(\phi) := \textstyle\int_{\mathbb{T}^n} \phi^{\bullet} f dm. \tag{4.1}$$

We must first establish the following:

Claim. *$\mathscr{L}_{[f]}$ is independent of the choice of representative of the coset $[f] \in {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A})$.*

Suppose $\widetilde{f}$ is some other representative of the coset $[f]$. Then, there exists a $g \in {}^{\perp}\mathscr{A}(\mathfrak{A})$ such that $\widetilde{f} = f + g$. By the definition of ${}^{\perp}\mathscr{A}(\mathfrak{A})$, we have:

$$\textstyle\int_{\mathbb{T}^n} \phi^{\bullet} \widetilde{f} dm = \int_{\mathbb{T}^n} \phi^{\bullet} f dm + \int_{\mathbb{T}^n} \phi^{\bullet} g \, dm = \int_{\mathbb{T}^n} \phi^{\bullet} f dm.$$

Since ϕ was chosen arbitrarily from $\mathscr{A}(\mathfrak{A})$, the claim follows.

Since $f \in {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}$, $\mathscr{L}_{[f]}$ vanishes on $\mathscr{I}_{\mathfrak{A},\mathbf{X}}$.

By Lemma 4.1, we can find functions $\Phi_1, \dots, \Phi_N \in \mathfrak{A}$ such that

$$\Phi_j(X_k) = \boldsymbol{\delta}_{jk}, \quad j, k = 1, \dots, N.$$

Set $a_j := \mathscr{L}_{[f]}(\Phi_j)$, $j = 1, \dots, N$. For each $\phi \in \mathscr{A}(\mathfrak{A})$, write

$$\widetilde{\phi} := \phi - \sum_{1 \leq j \leq N} \phi(X_j) \Phi_j,$$

which belongs to $\mathscr{I}_{\mathfrak{A},\mathbf{X}}$ (since the weak* closed ideals $\mathscr{I}_{\mathfrak{A},\mathbf{X}}$ and $\{\psi \in \mathscr{A}(\mathfrak{A}) : \psi(x) = 0 \ \forall x \in \mathbf{X}\}$ coincide). Thus

$$\begin{aligned} \mathscr{L}_{[f]}(\phi) &= \mathscr{L}_{[f]}\bigg(\sum_{1 \leq j \leq N} \phi(X_j) \Phi_j \bigg) \\ &= \sum_{1 \leq j \leq N} a_j \phi(X_j) \\ &= \sum_{1 \leq j \leq N} a_j \int_{\mathbb{T}^n} \phi^{\bullet} K(X_j, \cdot) dm \quad \forall \phi \in \mathscr{A}(\mathfrak{A}). \end{aligned} \tag{4.2}$$

In the last equality, we use the fact that $\phi^\bullet$ is the boundary-value function of a function in $H^\infty(\mathbb{D}^n)$ and, therefore, is in $H^2(\mathbb{T}^n)$. Then, (4.2) follows from the discussion in subsection 2.1. But note that the function

$$\sum_{1\leq j\leq N} a_j K(X_j,\boldsymbol{\cdot}) \in \mathbb{L}^1(\mathbb{T}^n)$$

itself belongs to ${}^\perp\mathscr{I}_{\mathfrak{A},\mathbf{X}}$. Thus, from (4.2), we see that f and $\sum_{1\leq j\leq N} a_j K(X_j,\boldsymbol{\cdot})$ differ by a function in ${}^\perp\mathscr{A}(\mathfrak{A})$. Hence the result. $\square$

The final result of this section is central to the proof of Theorem 1.3. At its heart is a close reading of the reason for the well-known fact that $A(\mathbb{D}^n)|_{\mathbb{T}^n}$ is approximating in modulus (see the paragraph following Result 1.2 for a definition). The class $\mathfrak{P}(R)$ below is as defined in the statement of Theorem 1.3.

Proposition 4.3. *Fix a positive integer $R \geq 1$. Let f be a positive, continuous function on $\mathbb{T}^n$. For each $\varepsilon > 0$, there exists a polynomial $p \in \mathfrak{P}(R)$ such that*

$$\sup_{\mathbb{T}^n} \big|f - |p|^2\big| < \varepsilon.$$

Proof. Let F_k denote the k-th Fejér kernel on $\mathbb{T}^n$ (i.e., the kernel associated to the Cesàro mean involving the characters parametrized by $(\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n$, $-k \leq \alpha_j \leq k$). Since f is positive and continuous, $\log(f)$ is continuous as well. By Fejér's theorem:

$$\log(f) * \mathsf{F}_k \longrightarrow \log f \;\text{ uniformly, as } k \to \infty. \tag{4.3}$$

By the properties of the Fejér kernels, $\log(f) * \mathsf{F}_k$ is a trigonometric polynomial and, as $\log(f)$ is real-valued, there exist polynomials $P_k \in \mathbb{C}[z_1,\dots,z_n]$ such that

$$\log(f) * \mathsf{F}_k(e^{i\theta_1},\dots,e^{i\theta_n}) \,=\, \mathsf{Re}\big(P_k(e^{i\theta_1},\dots,e^{i\theta_n})\big).$$

Let us now define $g_k : \mathbb{C}^n \longrightarrow \mathbb{C}$ by $g_k(z) := e^{P_k(z_1,\dots,z_n)/2}$, $z \in \mathbb{C}^n$. By (4.3) and the fact that $|e^A| = e^{\mathsf{Re}(A)}$ for any $A \in \mathbb{C}$, we get

$$\big|\, g_k|_{\mathbb{T}^n} \big|^2 \longrightarrow f \;\text{ uniformly, as } k \to \infty. \tag{4.4}$$

Let us now set

$$m := \max_{\zeta\in\mathbb{T}^n} f(\zeta), \quad \text{and} \quad M := \sqrt{2}\,\sqrt{(m+\varepsilon/2) + ((m+\varepsilon/2)^{1/2}+1)^2}.$$

For simplicity of notation, let us abbreviate $\sup_{\mathbb{T}^n}|\boldsymbol{\cdot}|$ to $\|\boldsymbol{\cdot}\|_{\mathbb{T}^n}$. By (4.4), there exists a positive integer k^ε such that

$$\big\|\, |g_k|^2 - f \big\|_{\mathbb{T}^n} < \varepsilon/2 \quad \forall k \geq k^\varepsilon. \tag{4.5}$$

Now set

$$\mu_{R,\varepsilon} := \min_{R\boldsymbol{\cdot}\overline{\mathbb{D}^n}} |g_{k^\varepsilon}| \quad \text{(which is a strictly positive number).}$$

The Taylor expansion of g_{k^ε}, the latter being entire, converges to g_{k^ε} uniformly on any fixed compact subset of $\mathbb{C}^n$. Thus, we can find a polynomial $p \in \mathbb{C}[z_1,\dots,z_n]$ such that

$$\sup_{R\boldsymbol{\cdot}\overline{\mathbb{D}^n}} |g_{k^\varepsilon} - p| < \min\left(\frac{\varepsilon}{2M}, \frac{\mu_{R,\varepsilon}}{2}, 1\right). \tag{4.6}$$

By our definition of $\mu_{R,\varepsilon}$, $p^{-1}\{0\} \cap (R\boldsymbol{\cdot}\overline{\mathbb{D}^n}) = \varnothing$. Hence, $p \in \mathfrak{P}(R)$.

Finally — making use of (4.6) — we estimate:

$$\begin{aligned} \|\,|g_{k^\varepsilon}|^2 - |p|^2\|_{\mathbb{T}^n} &\leq \|g_{k^\varepsilon} + p\|_{\mathbb{T}^n} \times \|g_{k^\varepsilon} - p\|_{\mathbb{T}^n} \\ &\leq \sqrt{2}\,\sqrt{\|g_{k^\varepsilon}\|^2_{\mathbb{T}^n} + \|p\|^2_{\mathbb{T}^n}}\;\frac{\varepsilon}{2M} \\ &\leq \varepsilon/2. \end{aligned}$$

By the above estimate and (4.5), we see that p is the desired polynomial. $\square$

5. The proof of Theorem 1.3

Before we give a proof of Theorem 1.3, it will be very useful to state a special case of Lemma 2.1 adapted to the situation that is of interest to us. The spaces of greatest interest to us are the quotient spaces:

$$\mathscr{A}(\mathfrak{A})/\mathscr{I}_{\mathfrak{A},\mathsf{X}} \quad \text{and} \quad {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathsf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A}), \tag{5.1}$$

these spaces being exactly as introduced in Section 4. Since this lemma will require some preliminary discussion, we divide this section into two subsections.

5.1. A few essential auxiliary lemmas.

We will need to work with a more general collection of objects than $\mathfrak{A}$. To this end — with $\mathfrak{A} \subseteq A(\mathbb{D}^n)$ as above — let I denote a *uniformly closed* ideal of $\mathfrak{A}$. Write

$$\mathscr{I} := \text{the weak* closure (in the sense of } (\bullet) \text{ in Section 3) of } \mathsf{I}.$$

As I is a subspace of $\mathfrak{A} \subset H^\infty(\mathbb{D}^n)$ we can, in view of Proposition 3.6 and the discussion at the beginning of Section 3, define:

$${}^{\perp}\mathscr{I} := \Big\{ f \in \mathbb{L}^1(\mathbb{T}^n) : \textstyle\int_{\mathbb{T}^n} \psi^\bullet f\, dm = 0 \text{ for each } \psi \in \mathscr{I} \Big\}.$$

With these notations, we have:

Lemma 5.1. *Let $\mathfrak{A}$ be a uniform subalgebra of $A(\mathbb{D}^n)$, and let I be a uniformly closed ideal in $\mathfrak{A}$. Then*

$${}^{\perp}\mathscr{I} = {}^{\perp}\mathsf{I} := \Big\{ f \in \mathbb{L}^1(\mathbb{T}^n) : \textstyle\int_{\mathbb{T}^n} \psi^\bullet f\, dm = 0 \textit{ for each } \psi \in \mathsf{I} \Big\}.$$

Proof. It is clear that ${}^{\perp}\mathscr{I} \subseteq {}^{\perp}\mathsf{I}$. Consider an arbitrary $\phi \in \mathscr{I}$. By (3.1) (we reiterate: owing to Proposition 3.6, $\phi \in H^\infty(\mathbb{D}^n)$), we have:

$$\phi(r\boldsymbol{\cdot})|_{\mathbb{T}^n} \longrightarrow \phi^\bullet \;\; m\text{-a.e. as } r \to 1^-.$$

Invoking Proposition 3.6 once more, there exists a sequence $\{\varphi_\nu\} \subset \mathsf{I}$ such that $\varphi_\nu \longrightarrow \phi$ uniformly on compact subsets of $\mathbb{D}^n$. Let us fix an $r \in (0,1)$. Then:

$$\varphi_\nu(r\boldsymbol{\cdot}) \longrightarrow \phi(r\boldsymbol{\cdot}) \;\text{ uniformly on each } \overline{D(0;\rho)}^{\,n},\ \rho \in (0,1).$$

By Proposition 3.6, $\phi(r\boldsymbol{\cdot}) \in A(\mathbb{D}^n) \cap \mathscr{I}$, and hence in I, for every $r \in (0,1)$. Since $\phi^\bullet \in \mathbb{L}^\infty(\mathbb{T}^n)$, we may apply the dominated convergence theorem to get:

$$0 = \lim_{r \to 1^-} \textstyle\int_{\mathbb{T}^n} \big(\, \phi(r\boldsymbol{\cdot})|_{\mathbb{T}^n} \big) g\, dm = \int_{\mathbb{T}^n} \phi^\bullet g\, dm \quad \forall g \in {}^{\perp}\mathsf{I}. \tag{5.2}$$

This establishes that ${}^{\perp}\mathsf{I} \subseteq {}^{\perp}\mathscr{I}$, and hence the result. □

The principal lemma of this subsection follows. But first, a few more words on our notation: we shall use $[\,\boldsymbol{\cdot}\,]$ to denote cosets in either of the two quotient spaces named in (5.1). However, we shall avoid ambiguity by using Greek letters when referring to cosets in $\mathscr{A}(\mathfrak{A})/\mathscr{I}_{\mathfrak{A},\mathsf{X}}$ and standard Roman italics when referring to cosets in ${}^{\perp}\mathscr{I}_{\mathfrak{A},\mathsf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A})$.

Lemma 5.2. *Let $\mathfrak{A}$ be a uniform subalgebra of $A(\mathbb{D}^n)$ and let $\mathsf{X} = \{X_1, \dots, X_N\}$, where the latter points are as in Theorem 1.3. For each $[\phi] \in \mathscr{A}(\mathfrak{A})/\mathscr{I}_{\mathfrak{A},\mathsf{X}}$, define*

$$L_{[\phi]}([f]) := \int_{\mathbb{T}^n} \phi^\bullet f\, dm \quad \forall [f] \in {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathsf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A}).$$

Then:

1. *$L_{[\phi]}([f])$ is independent of the choice of representatives of the cosets $[f] \in {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathsf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A})$ and $[\phi] \in \mathscr{A}(\mathfrak{A})/\mathscr{I}_{\mathfrak{A},\mathsf{X}}$. Furthermore, $L_{[\phi]}$ is an element of $({}^{\perp}\mathscr{I}_{\mathfrak{A},\mathsf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A}))^*$.*
2. *$\|[\phi]\| = \|L_{[\phi]}\|_{\rm op}$ for every $[\phi] \in \mathscr{A}(\mathfrak{A})/\mathscr{I}_{\mathfrak{A},\mathsf{X}}$.*

Proof. The proof of (1) is routine in view of the Claim in the proof of Proposition 4.2. Note that $L_{[\phi]}([f]) = \mathscr{L}_{[f]}(\phi)$ of Proposition 4.2. Thus, we already have a proof of the independence of $L_{[\phi]}([f])$ of the choice of the representative of the coset $[\phi]$.

The independence of the choice of representative of the coset $[f]$ follows from the definition of ${}^{\perp}\mathscr{A}(\mathfrak{A})$. That $L_{[\phi]} \in ({}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A}))^*$ is now routine.

To prove (2), we appeal to Lemma 2.1. We take

$$X = \mathbb{L}^1(\mathbb{T}^n), \quad S = {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}, \text{ and } E = {}^{\perp}\mathscr{A}(\mathfrak{A})$$

to get

$$\big({}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A})\big)^* \cong_{\text{isometric}} ({}^{\perp}\mathscr{A}(\mathfrak{A}))^{\perp}/({}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}})^{\perp}. \tag{5.3}$$

We now need to understand — in the notation of Lemma 2.1 — the coset $\Theta(L_{[\phi]})$. However, this will first require us to better understand the subspaces $({}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}})^{\perp}, ({}^{\perp}\mathscr{A}(\mathfrak{A}))^{\perp} \subset \mathbb{L}^{\infty}(\mathbb{T}^n)$. Recall the definitions of ${}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}$ and ${}^{\perp}\mathscr{A}(\mathfrak{A})$ — it follows from the $\mathbb{L}^1$–$\mathbb{L}^{\infty}$ duality that (see [17, Theorem 4.7], for instance):

$$({}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}})^{\perp} = \text{wk*}(\mathscr{I}_{\mathfrak{A},\mathbf{X}}, \mathbb{L}^1), \tag{5.4}$$

$$({}^{\perp}\mathscr{A}(\mathfrak{A}))^{\perp} = \text{wk*}(\mathscr{A}(\mathfrak{A}), \mathbb{L}^1). \tag{5.5}$$

By Lemma 3.3, we have

$$\text{wk*}(\mathscr{I}_{\mathfrak{A},\mathbf{X}}, \mathbb{L}^1) = \mathscr{I}^{\bullet}_{\mathfrak{A},\mathbf{X}} \quad \text{and} \quad \text{wk*}(\mathscr{A}(\mathfrak{A}), \mathbb{L}^1) = \mathscr{A}(\mathfrak{A})^{\bullet}. \tag{5.6}$$

From the above identities and the discussion at the beginning of Section 3, together with (5.3), (5.4) and (5.5), we deduce the useful fact:

$$\big({}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A})\big)^* \cong_{\text{isometric}} \mathscr{A}(\mathfrak{A})/\mathscr{I}_{\mathfrak{A},\mathbf{X}},$$

where the isometry is given by the isomorphism Θ described in Lemma 2.1.

Now, $\Theta(L_{[\phi]})$ is a coset in $\mathscr{A}(\mathfrak{A})/\mathscr{I}_{\mathfrak{A},\mathbf{X}}$, which we shall call $[\theta_{\phi}]$. As Θ is an isometry,

$$\|[\theta_{\phi}]\| = \|L_{[\phi]}\|_{\text{op}}. \tag{5.7}$$

Unravelling the construction of Θ (and by the manner in which a function in $\mathbb{L}^{\infty}(\mathbb{T}^n)$ induces a bounded linear functional of $\mathbb{L}^1(\mathbb{T}^n)$) we have that for any $F \in ({}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A}))^*$

$$F([f]) = L_{\Theta(F)}([f]) \quad \forall\,[f] \in {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}/{}^{\perp}\mathscr{A}(\mathfrak{A}).$$

Thus, if ϕ is any representative of $[\phi]$ and θ any representative of $[\theta_{\phi}]$, then:

$$L_{[\phi]}([f]) = \int_{\mathbb{T}^n} \theta^{\bullet}\,\widetilde{g}\,dm \quad \forall\,\widetilde{g} \in [f] \text{ and } \forall\,[f] \in {}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}}.$$

From this we infer that $(\theta^{\bullet} - \phi^{\bullet}) \in ({}^{\perp}\mathscr{I}_{\mathfrak{A},\mathbf{X}})^{\perp} = \mathscr{I}^{\bullet}_{\mathfrak{A},\mathbf{X}}$ by (5.6). But this means that $\|[\theta]\| = \|[\theta_{\phi}]\| = \|[\phi]\|$. Therefore, by (5.7) we have $\|[\phi]\| = \|L_{[\phi]}\|_{\text{op}}$. ☐

5.2. **A key proposition and Theorem 1.3.** We begin with a proposition that is the key result leading to the proof of Theorem 1.3. It gives us a way of linking a function ψ belonging to the dual algebra $\mathscr{A}$, that interpolates the data $\{(X_j, w_j) : 1 \leq j \leq N\}$, to conditions for $\sup_{\mathbb{D}^n} |\psi|$ to be ≤ 1. We shall continue to use the notation introduced in Sections 1 and 4, and extend the notation where needed. For instance

$$I^2_{\mathfrak{A},\mathbf{X}}(g) := \text{the closure of } I_{\mathfrak{A},\mathbf{X}}\big|_{\mathbb{T}^n} \text{ in } \mathbb{L}^2(\mathbb{T}^n, |g|^2 dm),$$

where $g \in \mathbb{L}^{\infty}(\mathbb{T}^n)$ and such that $|g| > c_g$ for some constant $c_g > 0$.

With those remarks, we can state and prove our key proposition.

Proposition 5.3. *Let $X_1, \dots, X_N$ be distinct points in $\mathbb{D}^n$. Let $\mathscr{A}$ be a weak* closed subalgebra of $H^\infty(\mathbb{D}^n)$ such that $\mathscr{A} = \mathscr{A}(\mathfrak{A})$ for some uniform subalgebra $\mathfrak{A} \subseteq A(\mathbb{D}^n)$ having a tame pre-annihilator. Fix an integer $R \geq 1$, and let $\mathfrak{P}(R) \subsetneq \mathbb{C}[z_1, \dots, z_n]$ be the class defined in Theorem 1.3. For any coset $[\phi] \in \mathscr{A}(\mathfrak{A})/\mathscr{I}_{\mathfrak{A},\mathbf{X}}$*

$$\|[\phi]\| \;=\; \sup\left\{ \left\|\Pi_{\mathfrak{A}^2(p)} \circ M_\phi^* \circ \Pi_{p,\mathbf{X}}\right\|_{\rm op} : p \in \mathfrak{P}(R)\right\}, \tag{5.8}$$

where

$$\Pi_{\mathfrak{A}^2(p)} := \text{the orthogonal projection of } \mathbb{L}^2(\mathbb{T}^n) \text{ onto } \mathfrak{A}^2(p),$$
$$\Pi_{p,\mathbf{X}} := \text{the orthogonal projection of } \mathfrak{A}^2(p) \text{ onto } \mathfrak{A}^2(p) \ominus I^2_{\mathfrak{A},\mathbf{X}}(p).$$

Proof. Lemma 5.2 suggests that to establish (5.8) we can work with the linear functionals $L_{[\phi]} \in ({}^\perp\mathscr{I}_{\mathfrak{A},\mathbf{X}}/{}^\perp\mathscr{A}(\mathfrak{A}))^*$. Let us fix a coset $[f]$. By Proposition 4.2, we can find constants $a_1, \dots, a_N \in \mathbb{C}$ — which depend only on the coset $[f]$ — such that

$$[f] \;=\; \Big[\sum_{1\leq j\leq N} a_j K(X_j, \boldsymbol{\cdot})\Big]. \tag{5.9}$$

In what follows (as well as in the next section), we shall use $\|\boldsymbol{\cdot}\|_1$ to denote the $\mathbb{L}^1$-norm on $\mathbb{L}^1(\mathbb{T}^n)$. Furthermore, $\|[f]\|$ will denote the quotient norm of $[f]$. Fix an $\varepsilon > 0$. It follows from (5.9) that there exists a function $G_\varepsilon \in {}^\perp\mathscr{A}(\mathfrak{A})$ such that

$$\Big\|\sum_{1\leq j\leq N} a_j K(X_j, \boldsymbol{\cdot}) + G_\varepsilon\Big\|_1 \;<\; \|[f]\| + \varepsilon.$$

(It is understood from (5.9) that the function $\sum_{1\leq j\leq N} a_j K(X_j, \boldsymbol{\cdot}) \in {}^\perp\mathscr{I}_{\mathfrak{A},\mathbf{X}}$ — this follows from the reproducing property of the Szegő kernel for $H^2(\mathbb{T}^n) \supset \mathscr{A}(\mathfrak{A})^\bullet$.) By Lemma 5.1 and the fact that $\mathfrak{A}$ has a tame pre-annihilator, we can find a function $H_\varepsilon \in (\mathcal{C}(\mathbb{T}^n;\mathbb{C}) \cap {}^\perp\mathscr{A}(\mathfrak{A}))$ such that $\|G_\varepsilon - H_\varepsilon\|_1 < \varepsilon$. Let us now write:

$$F_\varepsilon \;=\; \sum_{1\leq j\leq N} a_j K(X_j, \boldsymbol{\cdot}) + H_\varepsilon.$$

By (5.9) and the subsequent discussion, we have:

- (A) $[F_\varepsilon] = [f]$;
- (B) $F_\varepsilon \in \mathcal{C}(\mathbb{T}^n;\mathbb{C})$;
- (C) $\|F_\varepsilon\|_1 < \|[f]\| + 2\varepsilon$.

Recall that we have fixed an $R \geq 1$. Now, $|F_\varepsilon| + 3\varepsilon/4$ is a strictly positive continuous function on $\mathbb{T}^n$. Thus, by Proposition 4.3, there exists a polynomial $p^{(\varepsilon)} \in \mathfrak{P}(R)$ such that

$$|F_\varepsilon(\zeta)| + \varepsilon \;>\; |p^{(\varepsilon)}(\zeta)|^2 \;>\; |F_\varepsilon(\zeta)| + \varepsilon/2 \quad \forall \zeta \in \mathbb{T}^n. \tag{5.10}$$

In this paragraph, we shall take g to be any function in $A(\mathbb{D}^n)$ such that $g|_{\mathbb{T}^n}$ is non-vanishing. Write

$$F_g(\zeta) \;:=\; \overline{F_\varepsilon(\zeta)}/|g(\zeta)|^2 \quad \forall \zeta \in \mathbb{T}^n.$$

The projection operator $\Pi_{g,\mathbf{X}}$ will have a meaning analogous to $\Pi_{p,\mathbf{X}}$ defined above. In view of Lemma 3.3, it follows from a standard argument (see the first one-third of McCullough's argument for Proposition 5.9 in [15], for instance) that $\mathscr{A}(\mathfrak{A})^\bullet$ and $\mathscr{I}^\bullet_{\mathfrak{A},\mathbf{X}} \subset (\mathfrak{A}^2(1) \cap \mathbb{L}^\infty(\mathbb{T}^n))$. Thus, by the properties of g, we have:

$$\mathscr{A}(\mathfrak{A})^\bullet \subset \mathfrak{A}^2(g) \quad \text{and} \quad \mathscr{I}^\bullet_{\mathfrak{A},\mathbf{X}} \subset I^2_{\mathfrak{A},\mathbf{X}}(g). \tag{5.11}$$

We now compute:

$$\begin{aligned} L_{[\phi]}([f]) \;&=\; L_{[\phi]}([F_\varepsilon]) && \text{(by } (A) \text{ above)} \\ &=\; \langle \phi^\bullet, F_g\rangle_g \end{aligned}$$

$$
\begin{aligned}
&= \langle \Pi_{g,\mathbf{X}}(\phi^{\bullet}), F_g\rangle_g && \text{(by (5.11) and Lemma 5.2-(1))}\\
&= \langle \phi, \Pi_{g,\mathbf{X}}(F_g)\rangle_g\\
&= \langle 1, M_\phi^* \circ \Pi_{g,\mathbf{X}}(F_g)\rangle_g .
\end{aligned}
$$

Hence, we get the useful inequality:

$$
|L_{[\phi]}([f])| \;\leq\; \|\Pi_{\mathfrak{A}^2(g)} \circ M_\phi^* \circ \Pi_{g,\mathbf{X}}\|_{\rm op}\, \|1\|_g\, \|F_g\|_g, \tag{5.12}
$$

which holds true for *any* g with the properties stated above. Here $\|\cdot\|_g$ denotes the norm on $\mathfrak{A}^2(g)$.

At this stage, we shall take $g = p^{(\varepsilon)}$ in (5.12). Since $p^{(\varepsilon)} \in \mathfrak{P}(R)$, and $R \geq 1$, $p^{(\varepsilon)}$ has all the properties required of g in the previous paragraph. We ought to state that, after having chosen $g = p^{(\varepsilon)}$, the rest of the argument for this proof uses the same estimates that conclude the proof of [15, Theorem 5.13]. By (5.10), we have

$$
|F_{p^{(\varepsilon)}}(\zeta)| \;<\; 1 \quad \forall \zeta \in \mathbb{T}^n.
$$

Therefore, by the last inequality, (5.10) and (C) above, we have:

$$
\begin{aligned}
\|F_{p^{(\varepsilon)}}\|^2_{p^{(\varepsilon)}} &< \textstyle\int_{\mathbb{T}^n} |p^{(\varepsilon)}|^2\, dm \;<\; \| [f] \| + 3\varepsilon,\\
\|1\|^2_{p^{(\varepsilon)}} &= \textstyle\int_{\mathbb{T}^n} |p^{(\varepsilon)}|^2\, dm \;<\; \| [f] \| + 3\varepsilon.
\end{aligned}
$$

Combining the above inequalities with (5.12) and letting $\varepsilon \searrow 0$, we get:

$$
\frac{|L_{[\phi]}([f])|}{\| [f] \|} \;\leq\; \sup \big\{ \big\|\Pi_{\mathfrak{A}^2(p)} \circ M_\phi^* \circ \Pi_{p,\mathbf{X}}\big\|_{\rm op} : p \in \mathfrak{P}(R) \big\} \quad \text{if } [f] \neq [0].
$$

Since $[f]$ was chosen arbitrarily, the right-hand side of the above inequality actually dominates $\|L_{[\phi]}\|_{\rm op}$. We now appeal to Lemma 5.2 to get

$$
\|[\phi]\| \;\leq\; \sup \big\{ \big\|\Pi_{\mathfrak{A}^2(p)} \circ M_\phi^* \circ \Pi_{p,\mathbf{X}}\big\|_{\rm op} : p \in \mathfrak{P}(R) \big\}.
$$

The reverse inequality trivially holds true. This establishes (5.8). □

Finally, we present:

The proof of Theorem 1.3. Most of the steps of this proof are similar to those in the proofs of results analogous to Theorem 1.3 in the literature cited in Section 1. Hence, we shall be brief. We begin with two very standard facts. For each $p \in \mathfrak{P}(R)$.

- The set $\{\overline{K_{\mathfrak{A},p}}(X_1, \cdot), \dots, \overline{K_{\mathfrak{A},p}}(X_N, \cdot)\}$ spans $\mathfrak{A}^2(p) \ominus I^2_{\mathfrak{A},\mathbf{X}}(p)$.
- For any $\phi \in \mathscr{A}(\mathfrak{A})$, we have

$$
M_\phi^*(\overline{K_{\mathfrak{A},p}}(X_j, \cdot)) \;=\; \overline{\phi(X_j)}\, \overline{K_{\mathfrak{A},p}(X_j, \cdot)}, \quad j = 1, \dots, N.
$$

For any $f \in \mathfrak{A}^2(p) \ominus I^2_{\mathfrak{A},\mathbf{X}}(p)$, there exist $c_1, \dots, c_N \in \mathbb{C}$ such that

$$
f \;=\; \sum_{1 \leq j \leq N} c_j \overline{K_{\mathfrak{A},p}}(X_j, \cdot),
$$

whence, we compute:

$$
\Pi_{\mathfrak{A}^2(p)} \circ M_\phi^* \circ \Pi_{p,\mathbf{X}}(f) \;=\; \sum_{1 \leq j \leq N} c_j \overline{\phi(X_j)}\, \overline{K_{\mathfrak{A},p}(X_j, \cdot)}.
$$

From this it follows, *exactly* (and by an elementary computation) as in several of the works cited in Section 1 that:

$$
\begin{aligned}
&\big\|\Pi_{\mathfrak{A}^2(p)} \circ M_\phi^* \circ \Pi_{p,\mathbf{X}}\big\|_{\rm op} \;\leq\; 1\\
&\qquad \iff \Big[(1 - \phi(X_j)\overline{\phi(X_k)}) \big\langle K_{\mathfrak{A},p}(X_j, \cdot), K_{\mathfrak{A},p}(X_k, \cdot)\big\rangle_{\mathfrak{A}^2(p)}\Big]_{j,k=1}^N \;\geq\; 0. \qquad (5.13)
\end{aligned}
$$

Now, suppose that there exists a function $F \in \mathscr{A}(\mathfrak{A})$ such that $F(X_j) = w_j$ for each $j = 1, \dots, N$ and such that $\sup_{\mathbb{D}^n} |F| \leq 1$. This implies that $\|[F]\| \leq 1$. Then, by Proposition 5.3 and (5.13), (1.5) follows.

Conversely, assume (1.5). Let $\Phi_1, \dots, \Phi_N \in \mathfrak{A}$ be as given by Lemma 4.1. Write

$$\phi := \sum_{1 \leq j \leq N} w_j \Phi_j \in \mathfrak{A}.$$

Observe that $\phi(X_j) = w_j$ for $j = 1, \dots, N$. By (5.13) and Proposition 5.3, we get $\|[\phi]\| \leq 1$. From the latter we have, by definition:

$$\text{For each } \nu \in \mathbb{Z}_+, \ \exists \psi_\nu \in \mathscr{I}_{\mathfrak{A},\mathbf{X}} \text{ such that } \| \left.\phi\right|_{\mathbb{T}^n} + \psi_\nu^\bullet \|_\infty = \sup_{\mathbb{D}^n} |\phi + \psi_\nu| < 1 + 1/\nu.$$

By Montel's theorem, there exists a sequence $\nu_1 < \nu_2 < \nu_3 < \dots$ and a holomorphic function F defined on $\mathbb{D}^n$ such that

$$\phi + \psi_{\nu_k} \longrightarrow F \ \text{ uniformly on compact subsets of } \mathbb{D}^n \text{ as } k \to \infty.$$

By Proposition 3.6, $F \in \mathscr{A}(\mathfrak{A})$. Clearly $F(X_j) = w_j$ for $j = 1, \dots, N$, and $\sup_{\mathbb{D}^n} |F| \leq 1$. $\square$

6. The proof of Theorem 1.7

In this section, it will be assumed throughout that $n \geq 2$. Before we give a proof of Theorem 1.7, let us look at an explicit description of the space ${}^\perp H^\infty(\mathbb{D}^n)$. Write

$$\mathbb{Y}^n := \mathbb{Z}^n \setminus \mathbb{N}^n,$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$. Then, it is not hard to show that

$${}^\perp H^\infty(\mathbb{D}^n) = \text{the closure in } \mathbb{L}^1(\mathbb{T}^n) \text{ of } \operatorname{span}_{\mathbb{C}}\{ \left.\overline{z}_1^{\alpha_1} \, \overline{z}_2^{\alpha_2} \dots \overline{z}_n^{\alpha_n}\right|_{\mathbb{T}^n} : (\alpha_1, \dots, \alpha_n) \in \mathbb{Y}^n \} \tag{6.1}$$

(an argument for the above can be found in [15, Section 5]).

We can now present:

The proof of Theorem 1.7. We shall use notations analogous to those in Sections 4 and 5. Accordingly, we shall denote by $\mathscr{I}_{\mathbf{X}}$ the following ideal:

$$\mathscr{I}_{\mathbf{X}} := \text{the weak* closure of the set of all } A(\mathbb{D}^n)\text{-functions that vanish on } \mathbf{X},$$

where $\mathbf{X} = \{X_1, \dots, X_N\}$. We shall, in a very essential way, need to work with the spaces

$$H^\infty(\mathbb{D}^n)/\mathscr{I}_{\mathbf{X}} \quad \text{and} \quad {}^\perp\mathscr{I}_{\mathbf{X}}/{}^\perp H^\infty(\mathbb{D}^n).$$

The notation $\|[\psi]\|$, where $\psi \in H^\infty(\mathbb{D}^n)$, will have the same meaning as in Section 5. Similarly, $\boldsymbol{|}[f]\boldsymbol{|}$ will denote the quotient norm on ${}^\perp\mathscr{I}_{\mathbf{X}}/{}^\perp H^\infty(\mathbb{D}^n)$.

Let ϕ be an interpolant in $H^\infty(\mathbb{D}^n)$ for the given data. Since, by hypothesis, the data are extremal, we have

$$\|[\phi]\| = 1. \tag{6.2}$$

We appeal again to Lemma 5.2. Recall, yet again, the linear functional:

$$L_{[\phi]} : {}^\perp\mathscr{I}_{\mathbf{X}}/{}^\perp H^\infty(\mathbb{D}^n) \ni [f] \longmapsto \int_{\mathbb{T}^n} \phi^\bullet f dm.$$

By (6.2) and Lemma 5.2, we have $\|L_{[\phi]}\|_{\text{op}} = 1$. Furthermore, as $H^\infty(\mathbb{D}^n)/\mathscr{I}_{\mathbf{X}}$ is finite-dimensional, it follows from Lemma 5.2-(2) that

$$\exists f_0 \in {}^\perp\mathscr{I}_{\mathbf{X}} \text{ such that } \boldsymbol{|}[f_0]\boldsymbol{|} = 1 \text{ and } \int_{\mathbb{T}^n} \phi^\bullet f_0 dm = 1. \tag{6.3}$$

Step 1. *Finding "nice" coset-representatives for* $[f_0]$
By Proposition 4.2 — taking $\mathfrak{A} = A(\mathbb{D}^n)$, whence $\mathscr{A}(\mathfrak{A}) = H^\infty(\mathbb{D}^n)$ — there exist constants $a_1, \dots, a_N \in \mathbb{C}$, not all of which are 0, such that

$$[f_0] \;=\; \Big[\sum_{1\leq j\leq N} a_j K(X_j, \boldsymbol{\cdot})\Big].$$

(Recall that, by the reproducing property, $K(X_j, \boldsymbol{\cdot}) \in {}^{\perp}\mathscr{I}_{\mathsf{X}}$ for each $j = 1, \dots, N$.) By definition

$$\|[f_0]\| \;:=\; \inf\{\|f_0 + g\|_1 : g \in {}^{\perp}H^\infty(\mathbb{D}^n)\}.$$

So, if we fix $\varepsilon > 0$, there exists a function $g_\varepsilon \in {}^{\perp}H^\infty(\mathbb{D}^n)$ such that

$$\Big[\sum_{1\leq j\leq N} a_j K(X_j, \boldsymbol{\cdot}) + g_\varepsilon\Big] = [f_0] \quad \text{and} \quad 1 \leq \Big\|\sum_{j=1}^{N} a_j K(X_j, \boldsymbol{\cdot}) + g_\varepsilon\Big\|_1 < 1 + \varepsilon/2. \tag{6.4}$$

From the brief discussion prior to this proof, (6.1) in particular, it follows that there exists a polynomial P_ε, in z and $\overline{z}$, of the form

$$P_\varepsilon(z) \;=\; \sum_{\alpha\in\mathsf{F}(\varepsilon)} C_\alpha \overline{z}_1^{\alpha_1}\, \overline{z}_2^{\alpha_2} \dots \overline{z}_n^{\alpha_n},$$

where $\mathsf{F}(\varepsilon)$ is a finite subset of $\mathbb{Y}^n$, such that

$$\|\, P_\varepsilon|_{\mathbb{T}^n} - g_\varepsilon\|_1 \;<\; \varepsilon/2. \tag{6.5}$$

By the form of the polynomial P_ε, we see that $P_\varepsilon \in {}^{\perp}H^\infty(\mathbb{D}^n)$. Thus, by (6.4) and (6.5), we have

$$[f_0] \;=\; \Big[\sum_{1\leq j\leq N} a_j K(X_j, \boldsymbol{\cdot}) + P_\varepsilon|_{\mathbb{T}^n}\Big] \quad \text{and} \quad 1 \leq \Big\|\sum_{j=1}^{N} a_j K(X_j, \boldsymbol{\cdot}) + P_\varepsilon|_{\mathbb{T}^n}\Big\|_1 < 1 + \varepsilon. \tag{6.6}$$

Let us write

$$G_\varepsilon \;:=\; \sum_{1\leq j\leq N} a_j K(X_j, \boldsymbol{\cdot}) + P_\varepsilon|_{\mathbb{T}^n}\,.$$

Let us emphasise how regular G_ε is. Note, firstly, that for each $X_j \in \mathsf{X}$, $\overline{K(X_j, \boldsymbol{\cdot})}$ is holomorphic (in its second variable) in some neighbourhood — which depends on X_j — of $\overline{\mathbb{D}^n}$. Now define the function γ_ε which is holomorphic on $\mathsf{Ann}(0; 1\pm\delta)^n$ — where $\delta > 0$ is determined by $X_1, \dots, X_N$ — as follows:

$$\gamma_\varepsilon(z) \;:=\; \sum_{1\leq j\leq N} \overline{a}_j \overline{K(X_j, z)} + \sum_{\alpha\in\mathsf{F}(\varepsilon)} \overline{C}_\alpha \prod_{j=1}^{n} z^{\alpha_1} \quad \forall (z_1, \dots, z_n) \in \mathsf{Ann}(0; 1\pm\delta)^n.$$

Observe that

$$\overline{\gamma}_\varepsilon|_{\mathbb{T}^n} \;=\; G_\varepsilon. \tag{6.7}$$

In short, associated to $[f_0]$ is a family of coset-representatives G_ε that are restrictions to $\mathbb{T}^n$ of antiholomorphic functions and whose $\mathbb{L}^1$-norms decrease to 1.

Step 2. *Finding a sequence of measures with useful properties*
Since $\gamma_\varepsilon \in \mathcal{O}(\mathsf{Ann}(0; 1\pm\delta)^n)$, it follows from (6.7) that $G_\varepsilon^{-1}\{0\}$ is a real-analytic subset of $\mathbb{T}^n$. As $G_\varepsilon \not\equiv 0$, it follows from the basic theory of real-analytic sets that

$$m(G_\varepsilon^{-1}\{0\}) \;=\; 0 \quad \text{for each } \varepsilon > 0. \tag{6.8}$$

Let us now define the positive measures μ_ε on $\mathbb{T}^n$ such that $d\mu_\varepsilon = |G_\varepsilon|dm$. These measures have the following useful property:

$$\begin{aligned}\mu_\varepsilon\big[\{\zeta\in\mathbb{T}^n : 1-|\phi^\bullet(\zeta)|\geq\sqrt{\varepsilon}\}\big] \;&\leq\; \frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{T}^n}(1-|\phi^\bullet|)|G_\varepsilon|dm \\ &<\; \frac{1}{\sqrt{\varepsilon}}\left((1+\varepsilon)-\left|\int_{\mathbb{T}^n}\phi^\bullet G_\varepsilon dm\right|\right) \;=\; \sqrt{\varepsilon}, \end{aligned}\tag{6.9}$$

which follows from Chebyshev's inequality, (6.3) and (6.6).

We would ultimately like to estimate the *Lebesgue* measures of the above sets. To that end, we have the following observation. Write

$$\Gamma_\varepsilon(\zeta) \;:=\; \begin{cases}1/|G_\varepsilon(\zeta)|, & \text{if } \zeta\notin G_\varepsilon^{-1}\{0\},\\ 0, & \text{if } \zeta\in G_\varepsilon^{-1}\{0\}.\end{cases}$$

Clearly, $\Gamma_\varepsilon\in\mathbb{L}^1(\mathbb{T}^n, d\mu_\varepsilon)$ for each $\varepsilon>0$. It follows from (6.8) that

$$m(E) \;=\; \int_E \Gamma_\varepsilon d\mu_\varepsilon \quad \text{for every Lebesgue measurable set } E\subseteq\mathbb{T}^n \tag{6.10}$$

for each $\varepsilon>0$.

Step 3. *Completing the proof*

Recall that $\phi^\bullet$ is undefined on a set of Lebesgue measure zero. It will not affect the conclusions of the argument below if we fix $\phi^\bullet(\zeta)=0$ on the latter set. Note that

$$\{\zeta\in\mathbb{T}^n : 1-|\phi^\bullet(\zeta)|>0\} \;=\; \liminf_{k\to\infty}\{\zeta\in\mathbb{T}^n : 1-|\phi^\bullet(\zeta)|>1/k^3\}. \tag{6.11}$$

Denote the set on the left-hand side of the above equation as $\mathcal{S}$ and write $E_k := \{\zeta\in\mathbb{T}^n : 1-|\phi^\bullet(\zeta)|>1/k^3\}$. Let us define

$$\begin{aligned}A_k \;&:=\; \{\zeta\in E_k : |G_{1/k^6}(\zeta)|\geq 1/k\},\\ B_k \;&:=\; \{\zeta\in E_k : |G_{1/k^6}(\zeta)|< 1/k\}, \quad k=1,2,3,\dots\end{aligned}$$

From (6.9), we have $\mu_{1/k^6}(A_k)<1/k^3$. Thus, from (6.10), we get

$$m(A_k) \;=\; \int_{A_k}\frac{1}{|G_{1/k^6}|}d\mu_{1/k^6} \;\leq\; \frac{1}{k^2} \quad \forall k\in\mathbb{Z}_+. \tag{6.12}$$

Let us define $S_k := \{\zeta\in\mathbb{T}^n : |G_{1/k^6}(\zeta)|<1/k\}$. Then

$$S_k \;=\; \{\zeta\in\mathbb{T}^n : -\overline{G_{1/k^6}(\zeta)}G_{1/k^6}(\zeta) > -1/k^2\}, \quad k=1,2,3,\dots,$$

whence, by (6.7), each S_k is semi-analytic. And clearly, as $\|G_{1/k^6}\|_1\geq 1$, each S_k is a proper subset of $\mathbb{T}^n$.

Claim. $\mathcal{S}\subseteq\limsup_{k\to\infty}A_k\cup\liminf_{k\to\infty}S_k$.

Pick a $\zeta\in\mathcal{S}$. Then, $\exists k_1(\zeta)\in\mathbb{N}$ such that $\zeta\in E_k\ \forall k\geq k_1(\zeta)$. Suppose $\zeta\notin\limsup_{k\to\infty}A_k$. By definition, $\exists k_2(\zeta)\in\mathbb{N}$ such that $\zeta\notin A_k\ \forall k\geq k_2(\zeta)$. As A_k and B_k partition E_k, it follows that

$$\zeta\in B_k\subseteq S_k \quad \forall k\geq\max(k_1(\zeta),k_2(\zeta)).$$

The claim follows.

Recall that m is normalized to be a probability measure. Thus, by (6.12) and the Borel–Cantelli lemma, we have $m(\limsup_{k\to\infty}A_k)=0$. Finally, let us write:

$$N \;:=\; \limsup_{k\to\infty}A_k \quad\text{and}\quad S \;:=\; \liminf_{k\to\infty}S_k.$$

Since $A_k\cap S_k=\varnothing\ \forall k\in\mathbb{Z}_+$, it is very easy to see that $S\cap N=\varnothing$. Thus $\mathcal{S}\subset N\sqcup S$. $\square$

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