MA328 : INTRODUCTION TO SEVERAL COMPLEX VARIABLES
AUTUMN 2019

NOTE 1

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This note presents one of the implications in the big theorem characterizing domains of holomorphy. In the ordering of the statements presented in class, we give the proof of the implication (4) ⇒ (5). In other words:

Theorem. *Let $\Omega \subsetneq \mathbb{C}^n$ be a domain in $\mathbb{C}^n$ that admits a continuous plurisubharmonic exhaustion function. Then Ω admits a $\mathscr{C}^\infty$-smooth, strictly plurisubharmonic exhaustion function.*

Remark. The starting point of the above result is that the given exhaustion admits a sequence of smooth approximinants that are plurisubharmonic on an exhausting sequence of relatively-compact open subsets of Ω. In some sense, these approximinants must be "glued together" without destroying plurisubharmonicity. The **proof** of the above theorem, albeit slightly technical, presents one of the strategies for gluing together plurisubharmonic functions — in a sense dictated by the application at hand — so that plurisubharmonicity is preserved.

Proof. Let $E \in \mathcal{C}(\Omega) \cap \mathsf{psh}(\Omega)$ be the exhaustion function that is assumed to exist. Then,

$$\widetilde{E}(z) := E(z) + \|z\|^2, \quad z \in \Omega,$$

is also a plurisubharmonic exhaustion function. Here, $\|\cdot\|$ is the Euclidean norm. For any $\alpha \in \mathbb{R}$, write

$$\Omega_\alpha := \{z \in \Omega : \widetilde{E}(z) < \alpha\}.$$

By definition, $\Omega_\alpha \Subset \Omega_\beta$ whenever $\alpha < \beta$, and $\Omega_\alpha \Subset \Omega \ \forall \alpha \in \mathbb{R}$. By subtracting a large positive constant from $\widetilde{E}$ if necessary, we can assume without loss of generality that $\Omega_0 \neq \varnothing$. Also note that, by definition, $\widetilde{E}$ is bounded from below. Set $\mu := \inf_{z\in\Omega} \widetilde{E}(z)$.

We now define the sequence $\{\varepsilon_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}_+$ by:

$$\varepsilon_\nu := \mathsf{dist}[\,\Omega_{\nu+1}, \Omega^{\mathsf{C}}\,], \quad \nu = 0, 1, 2, \dots$$

Let χ be any smooth **polyradial** cut-off with $\mathsf{supp}(\chi) \subset \mathbb{B}^n$, where $\mathbb{B}^n$ is the open Euclidean unit ball centered at $0 \in \mathbb{C}^n$. If we write $\chi_\varepsilon := \varepsilon^{-2n}\chi(\boldsymbol{\cdot}/\varepsilon)$, $\varepsilon > 0$, then recall that the convolutions

$$\widetilde{E}_\varepsilon(z) := \chi_\varepsilon * \widetilde{E}(z), \;\; z \in \Omega^{(\varepsilon)} := \{w \in \Omega : \mathsf{dist}(z, \Omega^{\mathsf{C}}) > \varepsilon\},$$

are well-defined and have the following properties (refer to any reasonable treatment of the Lebesgue integral on Euclidean spaces):

a) $\widetilde{E}_\varepsilon \in \mathcal{C}^\infty(\Omega^{(\varepsilon)}) \cap \mathsf{psh}(\Omega^{(\varepsilon)})$ for each $\varepsilon > 0$.

b) $\widetilde{E}_\varepsilon \downarrow \widetilde{E}$ pointwise at each $z \in \Omega$ as $\varepsilon \to 0^+$.

For each $\nu \in \mathbb{N}$, *fix* some neighbourhood U_ν of $\overline{\Omega}_{\nu+1/2}$ such that $U_\nu \Subset \Omega_{\nu+1}$. Let $\Phi_\nu : \Omega \longrightarrow \mathbb{R}$, be any function satisfying the following two conditions:

- $\Phi_\nu(z) := \widetilde{E}_{\varepsilon_\nu}(z) + \|z\|^2$ for each $z \in U_\nu$,
- Φ_ν is a $\mathcal{C}^\infty$-extension of $\left(\widetilde{E}_{\varepsilon_\nu} + \|\cdot\|^2\right)\Big|_{U_\nu}$ to Ω.

By construction and (a), we have

$$\Phi_\nu|_{U_\nu} \in \mathsf{spsh}(U_\nu) \text{ for each } \nu \in \mathbb{N}. \tag{1}$$

Also, by (b), we have

$$\Phi_\nu(z) > \widetilde{E}(z) \;\; \forall z \in U_\nu \text{ and for each } \nu \in \mathbb{N}. \tag{2}$$

Let us now pick and **fix** a function $\kappa : \mathbb{R} \longrightarrow [0, \infty)$ that is $\mathcal{C}^\infty$-smooth and convex and has the following properties:

- $\kappa(x) = 0$ for each $x \leq 0$; and
- $\kappa'(x),\ \kappa''(x) > 0$ for each $x > 0$.

Notice that, owing to (2), we have the following inequality:

$$\Phi_\nu(z) - (\nu - 1/2) > 0 \quad \forall z \in (U_\nu \setminus \overline{\Omega}_{\nu-1/2}) \text{ and } \forall \nu \in \mathbb{Z}_+. \tag{3}$$

We will now construct inductively a sequence of smooth functions on Ω using the above Φ_ν's, each of which is strictly plurisubharmonic on an increasing sequence of subsets of Ω, which converges in a stronger sense than what (b) provides. To this end, it is easy to see that, owing to (3) and the properties of the function κ, we can find a constant $A_1 > 0$ that is so large that the function

$$\Psi_1 := \Phi_0 + A_1\, \kappa \circ (\Phi_1 - (1/2))$$

is strictly plurisubharmonic on $(U_1 \setminus \overline{\Omega}_{1/2})$ and such that $\Psi_1 > \widetilde{E}$ on the latter set. However, from these facts and by (1) and (2), we actually infer that Ψ_1 is strictly plurisubharmonic on U_1 and strictly dominates $\widetilde{E}$ on U_1.

Let us now make the following assumption:

$(*)$ We can find constants $A_1, \dots, A_m > 0$, $m \in \mathbb{Z}_+$, such that the function

$$\Psi_m := \Phi_0 + \sum_{j=1}^{m} A_j\, \kappa \circ (\Phi_j - (j - 1/2))$$

is strictly plurisubharmonic on U_m and $\Psi_m(z) > \widetilde{E}(z)$ for every $z \in U_m$.

By $(*)$ and by precisely the same argument as above, we can infer that the statement obtained by replacing m by $(m+1)$ in $(*)$ is true. Hence, by induction, $(*)$ is true for each $m \in \mathbb{Z}_+$.

Claim. *The sequence $\{\Psi_\nu\}$ converges uniformly on compact subsets of Ω.*
To see this, let us fix a compact $K \subset \Omega$. Let

- $\nu_K \in \mathbb{N}$ be such that $K \subset \Omega_{\nu+1/2}\ \forall \nu \geq \nu_K$.
- $\nu_K^* \in \mathbb{Z}_+$ be such that

  $$\bigcup_{z \in K} (z + \varepsilon_j \mathbb{B}^n) \subset \Omega_\nu \quad \forall \nu \geq \nu_K^* \text{ and } \forall j \geq \nu_K.$$

  (This is possible because $\varepsilon_j > \varepsilon_{j+1}\ \forall j \in \mathbb{N}$.)
- $M_K := \sup_{z \in K} \|z\|^2$.

We can now estimate:

$$\begin{aligned}
|\Phi_j(z)| &\leq \left|\int_{\mathbb{C}^n} \widetilde{E}(z-\varepsilon_j w)\,\chi(w)\,dV(w)\right| + M_K \\
&\leq \int_{\mathbb{B}^n} |\widetilde{E}(z-\varepsilon_j w)|\,dV(w) + M_K \\
&\leq \operatorname{vol}(\mathbb{B}^n)\max\left(\nu_K^*,\,|\mu|\,\right) + M_K \quad \forall z\in K \;\text{ and }\; \forall j\geq \nu_K.
\end{aligned}$$

Observe that the last bound depends **only** on K. Thus, we can find an integer $\widetilde{\nu}_K \geq \nu_K$ such that

$$\kappa\circ(\Phi_j - (j-1/2))(z) \,=\, 0 \quad \forall z\in K \;\text{ and }\; j\geq \widetilde{\nu}_K.$$

Thus, the sequence $\{\Psi_\nu\}$ saturates on compacts. This certainly establishes the above claim. But because it saturates, it establishes *a fortiori* the following:

Fact. *Write* $\Psi := \lim_{\nu\to\infty}\Psi_\nu$*. Then,* Ψ *is* $\mathcal{C}^\infty$*-smooth and strictly plurisubharmonic on* Ω*.*

One can now complete the proof easily. It follows from the fact that $(*)$ is true for each $m\in\mathbb{Z}_+$ that $\Psi\geq\widetilde{E}$. Thus

$$\{z\in\Omega : \Psi(z)\leq\alpha\} \,\subseteq\, \overline{\Omega}_\alpha \,\subsetneq\, \Omega \quad \forall\alpha\in\mathbb{R}.$$

Since each $\overline{\Omega}_\alpha$ is, by definition, compact, we conclude that Ψ is an exhaustion function. □