MA 328 : INTRODUCTION TO SEVERAL COMPLEX VARIABLES

AUTUMN 2019

HOMEWORK 3

Instructor: GAUTAM BHARALI **DUE: Thursday, Nov. 21, 2019**

Note:

a) You are allowed to discuss these problems with your classmates, but individually-written and **original** write-ups are expected for submission. Please **acknowledge** any persons from whom you received help in solving these problems.

b) Given a multi-index $\alpha \in \mathbb{N}^n$, we shall use the following notation:

$$|\alpha| := \alpha_1 + \cdots + \alpha_n\,,$$
$$\alpha! := \alpha_1! \ldots \alpha_n!\,,$$
$$z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n}\,.$$

c) The notation $I \in \mathscr{I}_q$ means that I is an increasing q-tuple in $\{1, \ldots, n\}^q$.

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1. Let Ω be a non-empty open subset of $\mathbb{R}^N$, $N \geq 2$, and $u : \Omega \longrightarrow [-\infty, +\infty)$ be an upper-semicontinuous function. Show that u is subharmonic if and only if for each open Euclidean ball $B \Subset \Omega$, given any function h that is harmonic on an open ball B^* concentric to B such that $\overline{B} \subsetneq B^* \subseteq \Omega$ and satisfies $h|_{\partial B} \geq u|_{\partial B}$, we have

$$h(x) \geq u(x) \quad \forall x \in B.$$

Hint. Consider an appropriate topological consequence of upper-semicontinuity.

2. Let Ω be a non-empty open subset of $\mathbb{C}$. Let A be a non-empty set and suppose the family $\{u_\alpha\}_{\alpha \in A} \subset \mathsf{subh}(\Omega)$. Assume that for each $\alpha \in A$, $u_\alpha \not\equiv -\infty$ on each connected component of Ω. If the function

$$U(z) := \sup_{\alpha \in A} u_\alpha(z) \quad \forall z \in \Omega$$

is upper-semicontinuous, then show that $U \in \mathsf{subh}(\Omega)$.

Note. The condition on the behaviour of u_α on each of the connected components of Ω is not required.

3. Let Ω be a non-empty open subset of $\mathbb{C}^n$ and let U be plurisubharmonic on Ω. Prove that for each $a \in \Omega$, $U(a) = \limsup_{z \to a} U(z)$.

4. *This problem is meant to demonstrate an analytical characterization of convexity of smoothly-bounded domains.* To this end, consider a domain $\Omega \subsetneq \mathbb{R}^N$, $N \geq 2$, with $\mathcal{C}^2$-smooth boundary, and provide details for the following outline. **Fix** a defining function ϱ of class $\mathcal{C}^2$. Let $p \in \partial\Omega$.

(*a*) Describe an explicit affine change of coordinate $\tau : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ such that $\tau(p) = 0$ and, using the notation

$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} := \tau \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix},$$

we have $D\tau(p)\big(\nabla\varrho(p)\big) = \|\nabla\varrho(p)\|(0, \ldots, 0, -1)$ and $D\tau(p) : T_p(\partial\Omega) \longrightarrow \{y \in \mathbb{R}^N : y_N = 0\}$.

(b) Re-express the equation of $\partial\Omega$ in $(y_1, \dots, y_N)$-coordinates: i.e., show that there exists an open ball B centred at $y = 0$ and a function $\varphi : \big(B \cap \{y : y_N = 0\}\big) \longrightarrow \mathbb{R}$ such that

$$\partial(\tau(\Omega)) \cap B \;=\; \{y \in B : y_N = \varphi(y_1, \dots, y_{N-1})\}.$$

(c) From (a) and (b), deduce that Ω is convex if and only if

$$\sum_{j,k=1}^{N} \frac{\partial^2 \varrho}{\partial x_j \partial x_k}(p) V_j V_k \;\geq\; 0 \quad \forall V \in T_p(\partial\Omega) \text{ and } \forall p \in \partial\Omega.$$

Hint. You may use **without** proof the fact that affine maps preserve convexity.

5. Let Ω be a non-empty open subset of $\mathbb{C}$ and let u be subharmonic on Ω. Let $\kappa : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing convex function. Define

$$U(z) \;:=\; \begin{cases} \kappa \circ u(z), & \text{if } z \notin u^{-1}\{-\infty\}, \\ \lim_{x \to -\infty} \kappa(x), & \text{if } z \in u^{-1}\{-\infty\}. \end{cases}$$

Show that U is subharmonic

Hint. You **need not** establish that the limit stated above exists. Look through the literature for a suitable theorem that involves convex functions and finite measures, and **state clearly** any such theorem that you use.

6. Let X be an n-dimensional complex manifold and let $\mathfrak{A} = \{(U_\alpha, \psi_\alpha) : \alpha \in A\}$ be the complex structure on X. Let TX be the classical (i.e., real) tangent bundle of X obtained by viewing $\mathfrak{A}$ as a $\mathcal{C}^\infty$-smooth atlas. Recall that this means that — denoting by π the bundle-projection of TX onto X — we have:

- for each $\alpha \in A$, homeomorphisms h_α such that the diagrams

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\;h_\alpha\;} & U_\alpha \times \mathbb{R}^{2n} \\ & \searrow^{\pi} & \big\downarrow \mathsf{proj}_1 \\ & & U_\alpha \end{array}$$

  commute (here proj_1 denotes the projection onto the first factor); and

- smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(2n, \mathbb{R})$, for all $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, that determine the transition functions for TX;

that are canonically determined by the collection $\{\psi_\alpha : \alpha \in A\}$. Write

$$T^{\mathbb{C}}X \;:=\; \bigcup\nolimits_{x \in X} (T_x X) \otimes \mathbb{C}.$$

Show, using $\mathfrak{A}$, that $T^{\mathbb{C}}X$ can be endowed with the structure of a smooth complex vector bundle whose fibres are $2n$-dimensional complex vector spaces.

7. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces and let $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a densely-defined unbounded (linear) operator. Show that T is a closed operator if and only if $\mathsf{Dom}(T)$ is a Hilbert space when equipped with the graph norm.

8. Let $q \geq 1$. Recall that the *formal adjoint* of $\overline{\partial} : \mathbb{L}^2_{(0,q-1)}(\Omega;\phi_1) \longrightarrow \mathbb{L}^2_{(0,q)}(\Omega;\phi_2)$, (where $\overline{\partial}$ is defined in the sense of distributions) is the adjoint of $\overline{\partial}$ on smooth $(0,q)$-forms "paired against $(\mathcal{C}^\infty_{\mathsf{c}})^{0,q-1}(\Omega)$." Given $\alpha \in \mathscr{I}_{q-1}$ and $\beta \in \mathscr{I}_q$, define

$$\varepsilon^{j\beta}_{\alpha} := \begin{cases} 0, & \text{if } j \in \alpha, \\ 0, & \text{if } \{j\} \cup \alpha \neq \beta, \\ & \\ \operatorname{sgn}\begin{pmatrix} j & \alpha_1 & \dots & \alpha_{q-1} \\ \beta_1 & \beta_2 & \dots & \beta_q \end{pmatrix}, & \text{if } \{j\} \cup \alpha = \beta, \end{cases}$$

where $1 \leq j \leq n$. Denoting by $\overline{\delta}^*$ formal adjoint of $\overline{\partial} : \mathbb{L}^2_{(0,q-1)}(\Omega;\phi_1) \longrightarrow \mathbb{L}^2_{(0,q)}(\Omega;\phi_2)$, show that

$$\overline{\delta}^*_{q-1}\left(\sum_{\beta\in\mathscr{I}_q} f_\beta d\overline{z}^\beta\right) = \sum_{\alpha\in\mathscr{I}_{q-1}}\left\{\sum_{\beta\in\mathscr{I}_q}\sum_{j=1}^{n} \varepsilon^{j\beta}_{\alpha} e^{\phi_1-\phi_2}\left(f_\beta\frac{\partial\phi_2}{\partial z_j} - \frac{\partial f_\beta}{\partial z_j}\right)\right\} d\overline{z}^\alpha,$$

where $\sum_{\beta\in\mathscr{I}_q} f_\beta d\overline{z}^\beta \in (\mathcal{C}^\infty)^{0,q}(\Omega)$.