MA 328 : INTRODUCTION TO SEVERAL COMPLEX VARIABLES
AUTUMN 2019
HOMEWORK 2

Instructor: GAUTAM BHARALI **DUE: Saturday, Oct. 5, 2019**

Note:

a) You are allowed to discuss these problems with your classmates, but individually-written and **original** write-ups are expected for submission. Please **acknowledge** any persons from whom you received help in solving these problems.

b) Given a multi-index $\alpha \in \mathbb{N}^n$, we shall use the following notation:

$$|\alpha| := \alpha_1 + \dots + \alpha_n \,,$$
$$\alpha! := \alpha_1! \dots \alpha_n! \,,$$
$$z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n} \,.$$

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1. Show that the $\overline{\partial}$-problem

$$\frac{\partial u}{\partial \overline{z}} = \phi,$$

where $\phi \in \mathcal{C}^1_{\mathsf{c}}(\mathbb{C})$ does **not** necessarily have a compactly-supported solution.
Hint. First consider the solution

$$u(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(w)}{w - z} dA(w),$$

for an appropriately chosen ϕ.

2. Provide details for the outline below to show that any open set $\Omega \subsetneq \mathbb{C}$ is a domain of holomorphy:

a) Construct a sequence $\{a_\nu\}_{\nu \in \mathbb{N}} \subset \Omega$ that has no limit points in Ω and such that $\overline{\{a_\nu : \nu \in \mathbb{N}\}} \setminus \{a_\nu : \nu \in \mathbb{N}\} = \partial\Omega$.

b) State **clearly** a suitable theorem from the function theory in one complex variable to construct a function $\varphi \in \mathcal{O}(\Omega)$ such that $\varphi \not\equiv 0$ and $\varphi(a_\nu) = 0$ for $\nu = 0, 1, 2, \dots$

c) Show that it is **impossible** to find any pair of open sets (U, V) such that $\varnothing \neq U \subset V \cap \Omega$, V is connected, and $V \nsubseteq \Omega$, such that $\varphi|_U$ extends to function $F_\varphi \in \mathcal{O}(V)$.

3. The *Hartogs triangle* is the domain in $\mathbb{C}^2$ given by:

$$\Omega := \{(z, w) \in \mathbb{C}^2 : |z| < |w| < 1\}.$$

Show that Ω is a domain of holomorphy. For $\varepsilon > 0$, the set $\cup_{z \in \overline{\Omega}} B^n(z; \varepsilon)$ — where $B^n(z; \varepsilon)$ denotes the open Euclidean ball of radius ε with centre z — is called an *ε-neighbourhood* of $\overline{\Omega}$. Show that for $\varepsilon > 0$ sufficiently small, no ε-neighbourhood of $\overline{\Omega}$ is a domain of holomorphy.

4. Consider the function $f(z, w) := \frac{1}{1-(z+w)}$.

a) Find the power-series development of f in some small neighbourhood of $(0,0) \in \mathbb{C}^2$.

b) Find the domain of convergence of the series that you computed in part (*a*).

5. Let S denote the power series $\sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$ and let $\mathscr{C}(S)$ denote it domain of convergence. Let

$$\Lambda\big(\mathscr{C}(S)\big) := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in \mathscr{C}(S)\}.$$

Show that:

a) $\Lambda\big(\mathscr{C}(S)\big)$ is open.

b) If $z \in \mathscr{C}(S)$, then there exists an $x \in \Lambda\big(\mathscr{C}(S)\big)$ such that $|z_j| \leq e^{x_j}$ for $j = 1, \dots, n$.

6. Let $n \geq 2$, $0 < r < 1$, and write

$$\Omega := \big(D(0;r) \times \mathbb{D}^{n-1}\big) \cup \big(\mathbb{D}^{n-1} \times D(0;r)\big).$$

Describe **explicitly**, in terms of r, a Reinhardt domain $\widetilde{\Omega} \supsetneq \Omega$ such that for each $f \in \mathcal{O}(\Omega)$, there exists $F_f \in \mathcal{O}(\widetilde{\Omega})$ such that $F_f|_\Omega = f$.

7. Let Ω_1 and Ω_2 be domains in $\mathbb{C}^n$ and let $F : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphism of Ω_1 onto Ω_2. Show that if Ω_1 is a domain of holomorphy, then so is Ω_2.

8. Let $\Omega_j \subseteq \mathbb{C}^{n_j}$, $j = 1, 2$, be domains of holomorphy. Show that the open set $\Omega_1 \times \Omega_2$ is a domain of holomorphy.

9. Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a holomorphic map, let $\Omega \subsetneq \mathbb{C}^n$ be a domain of holomorphy, and suppose $F^{-1}(\Omega)$ is bounded. Prove that $F^{-1}(\Omega)$ is a domain of holomorphy.