PEAK-INTERPOLATING CURVES FOR $A(\Omega)$ FOR FINITE-TYPE DOMAINS IN $\mathbb{C}^2$

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Abstract. Let Ω be a bounded, weakly pseudoconvex domain in $\mathbb{C}^2$, having smooth boundary. $A(\Omega)$ is the algebra of all functions holomorphic in Ω and continuous upto the boundary. A smooth curve $\boldsymbol{C} \subset \partial\Omega$ is said to be complex-tangential if $T_p(\boldsymbol{C})$ lies in the maximal complex subspace of $T_p(\partial\Omega)$ for each $p \in \boldsymbol{C}$. We show that if $\boldsymbol{C}$ is complex-tangential and $\partial\Omega$ is of constant type along $\boldsymbol{C}$, then every compact subset of $\boldsymbol{C}$ is a peak-interpolation set for $A(\Omega)$. Furthermore, we show that if $\partial\Omega$ is real-analytic and $\boldsymbol{C}$ is an arbitrary real-analytic, complex-tangential curve in $\partial\Omega$, compact subsets of $\boldsymbol{C}$ are peak-interpolation sets for $A(\Omega)$.

1. Statement of the main result

Let Ω be a bounded domain in $\mathbb{C}^n$, and let $A(\Omega)$ be the algebra of functions continuous on $\overline{\Omega}$ and holomorphic in Ω. Recall that a compact subset $K \subset \partial\Omega$ is called a **peak-interpolation set for $\boldsymbol{A(\Omega)}$** if given any $f \in \mathcal{C}(K)$, $f \not\equiv 0$, there exists a function $F \in A(\Omega)$ such that $F|_K = f$ and $|F(\zeta)| < \sup_K |f|$ for every $\zeta \in \overline{\Omega} \setminus K$.

We are interested in determining when a smooth submanifold $\boldsymbol{M} \subset \partial\Omega$ is a peak-interpolation set for $A(\Omega)$. When Ω is a strictly pseudoconvex domain having $\mathcal{C}^2$ boundary, and $\boldsymbol{M}$ is of class $\mathcal{C}^2$, the situation is very well understood; refer to the works of Henkin & Tumanov [9], Nagel [10], and Rudin [13]. In the strictly pseudoconvex setting, $\boldsymbol{M}$ is a peak-interpolation set for $A(\Omega)$ if and only if $\boldsymbol{M}$ is **complex-tangential**, i.e. $T_p(\boldsymbol{M}) \subset H_p(\partial\Omega)\ \forall p \in \boldsymbol{M}$. Here, and in what follows, for any submanifold $\boldsymbol{M} \subseteq \partial\Omega$, $T_p(\boldsymbol{M})$ will denote the real tangent space to $\boldsymbol{M}$ at the point $p \in \boldsymbol{M}$, while $H_p(\partial\Omega)$ will denote the maximal complex subspace of $T_p(\partial\Omega)$.

Very little is known, however, when Ω is a weakly pseudoconvex of finite type (There are several notions of type for domains in $\mathbb{C}^n$, $n \geq 2$, but they all coincide for pseudoconvex domains in $\mathbb{C}^2$. See Section 2 below.). In view of a result by Henkin & Tumanov [9], or a similar result by Nagel & Rudin [11], it is still necessary for $\boldsymbol{M}$ to be complex-tangential. However, showing even that any smooth compact complex-tangential arc in $\partial\Omega$ is a peak-interpolation set for $A(\Omega)$, for a general smoothly bounded weakly pseudoconvex domain of finite type, is a difficult problem. This is because doing so would necessarily imply that every point in $\partial\Omega$ is a peak point for $A(\Omega)$. Whether or not this is true for general pseudoconvex domains of finite type is an extremely difficult open question in the theory of functions in several complex variables, but this fact is certainly known for smoothly

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bounded finite type domains in $\mathbb{C}^2$ – see [1] of Bedford & Fornaess, [6] of Fornaess & McNeal and [7] of Fornaess & Sibony (and we will use this fact in one of our results below). In this paper we show, among other things, that when Ω is a bounded domain in $\mathbb{C}^2$, $\partial\Omega$ is real-analytic and $\boldsymbol{C} \subset \partial\Omega$ is a real-analytic curve, it suffices that $\boldsymbol{C}$ be complex-tangential for every compact subset of $\boldsymbol{C}$ to be a peak-interpolation set for $A(\Omega)$.

More precisely, our main result is as follows :

Theorem 1.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ having smooth boundary, and let $\boldsymbol{C} \subset \partial\Omega$ be a smooth curve.*

(i) *Let $\partial\Omega$ be of class $\mathcal{C}^\infty$ and Ω be of finite type. If $\boldsymbol{C}$ is complex-tangential, and if $\partial\Omega$ is of constant type along $\boldsymbol{C}$, then each compact subset of $\boldsymbol{C}$ is a peak-interpolation set for $A(\Omega)$.*

(ii) *Let Ω have real-analytic boundary, and let $\boldsymbol{C} \subset \partial\Omega$ be a real-analytic, complex-tangential curve. Then, each compact subset of $\boldsymbol{C}$ is a peak interpolation set for $A(\Omega)$.*

Observe that in (ii) above, we do *not* assume that $\partial\Omega$ is of constant type along $\boldsymbol{C}$.

2. Some notation and introductory remarks

We begin by defining the notion of type.

Definition 2.1. Let $\Omega \subset \mathbb{C}^2$ be a bounded domain having a smooth boundary. Let $p \in \partial\Omega$. The **type** of p, denoted by $\tau(p)$, is the maximum order of contact that the germ of a 1-dimensional complex variety through p can have with $\partial\Omega$ at p. The point p is said to be of finite type if $\tau(p) < \infty$. The domain Ω is said to be of **finite type** if there is an $N \in \mathbb{N}$ such that $\tau(p) \le N$ for each $p \in \partial\Omega$.

Remark 2.2. Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. Suppose $p \in \partial\Omega$, $\tau(p) = N$ and that there exist local holomorphic coordinates $(U; \zeta_1, \zeta_2)$, near p, with respect to which $p = 0$ and with respect to which $(U \cap \partial\Omega)$ is defined by

$$\rho(\zeta) = A(\zeta_1) + O(v_2^2, |\zeta_1||v_2|) - u_2, \tag{2.1}$$

where $\zeta_k := u_k + iv_k$, and $A(\zeta_1) = O(|\zeta_1|^2)$, then

(1) N is the leading order in ζ_1 of A.

(2) Owing to the pseudoconvexity of Ω, N is an even number.

The above are the consequences of a computation on smoothly bounded pseudoconvex domains in $\mathbb{C}^2$ of finite type at $p \in \partial\Omega$, which is given in [8, Lecture 28]. Examining this calculation, we can infer that

(3) Suppose $\varPhi = (\phi_1, \phi_2) : (U, p) \to (\mathbb{C}^2, 0)$ is a smooth change of coordinate – and write $\zeta_j = \phi_j(z_1, z_2)$, $j = 1, 2$ – such that $\overline{\partial}\phi_j$, $j = 1, 2$, vanishes to infinite order at p, and such that $(U \cap \partial\Omega)$ (with respect to these new coordinates) has a defining function of the form (2.1). Then, conclusions (1) and (2) above continue to hold.

We now present some notation. For a $\mathcal{C}^2$ function ϕ defined in some open set in $\mathbb{C}^n$, we will adopt the following notation

$$\partial_j\phi = \frac{\partial\phi}{\partial z_j}\ , \qquad \partial_{\bar{j}}\phi = \frac{\partial\phi}{\partial \overline{z}_j}\ ,$$

$$\partial^2_{jk}\phi = \frac{\partial^2\phi}{\partial z_j\partial z_k}\ , \quad \partial^2_{j\bar{k}}\phi = \frac{\partial^2\phi}{\partial z_j\partial \overline{z}_k}\ , \quad \partial^2_{\bar{j}\bar{k}}\phi = \frac{\partial^2\phi}{\partial \overline{z}_j\partial \overline{z}_k}\ .$$

And if F is a smooth function defined in a neighbourhood of $0 \in \mathbb{R}^N$, we define (borrowing our notation from [3])

$$\begin{aligned}\mathrm{In}(F) &:= \text{ the leading homogeneous polynomial in the Taylor expansion of } F \text{ around } 0,\\ \mathrm{ord}(F) &:= \text{ the degree of } \mathrm{In}(F).\end{aligned}$$

In what follows, $B(p;r)$ will denote the open Euclidean ball in $\mathbb{C}^2$ centered at $p \in \mathbb{C}^2$ and having radius r, while $D(a;r)$ will denote the open disc in $\mathbb{C}$ centered at $a \in \mathbb{C}$ and having radius r. Several parameters occur in our analysis and the independence of the quantitative estimates – occuring in the results below – from these parameters will be of some concern. We will express such estimates via the notation $X \lesssim Y$ – meaning that there is a constant $C > 0$, independent of all parameters, so that $X \leq CY$.

A standard approach to proving that $\boldsymbol{C} \subset \partial\Omega$ is a peak-interpolation set – $\boldsymbol{C}$, $\partial\Omega$ smooth – which is encountered in the papers [9] and [13], makes use of Bishop's theorem [2], which we now state :

Theorem (Bishop)**.** *Let Ω be a bounded domain in $\mathbb{C}^n$. A compact subset $K \subset \partial\Omega$ is a peak-interpolation set for $A(\Omega)$ if and only if for every annihilating measure $\mu \perp A(\Omega)$, $|\mu|(K) = 0$.*

In the above theorem, an **annihilating measure** refers to a regular, complex Borel measure on $\overline{\Omega}$ which, viewed as a bounded linear functional on $\mathcal{C}(\overline{\Omega})$, annihilates $A(\Omega)$. A variation of the aforementioned approach – needed in the proof of our main theorem – involves showing that if for any $p \in \boldsymbol{C}$ there is a small neighbourhood $V_p \ni p$ such that for each bump function $\chi \in \mathcal{C}^\infty_c(V_p;[0,1])$ with $rm \int[\chi^{-1}\{1\}] \cap \boldsymbol{C}$ being an open arc in $\boldsymbol{C}$, there is a sequence of functions $\{h_k\}_{k\in\mathbb{N}}$ such that : (i) $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ and is uniformly bounded on $\overline{\Omega}$, (ii) $\lim_{k\to\infty} h_k(z) = 0\ \forall z \in \overline{\Omega} \setminus (\boldsymbol{C} \cap V_p)$, and (iii) $\lim_{k\to\infty} h_k(z) = \chi(z)\ \forall z \in \boldsymbol{C} \cap V_p$. We explain in the next section why Theorem 1.1-(i) follows from the existence of such a $\{h_k\}_{k\in\mathbb{N}}$.

The key step in our proof is to show that if $\boldsymbol{C}$ is as described in Theorem 1.1-(i), then for each $p \in \boldsymbol{C}$ we can find a small neighbourhood $V_p \ni p$ so that for any $U \Subset V_p$, for which $\boldsymbol{C} \cap U$ is an arc, there is a smooth function G in V_p which is almost holomorphic with respect to $\boldsymbol{C} \cap V_p$ and peaks on $\boldsymbol{C} \cap \overline{U}$. Further, one requires that this almost holomorphic peak function must approach the value 1 at a controlled rate. We show that

$$|G(z)| \ \leq\ 1 - C\mathrm{dist}[z, \boldsymbol{C} \cap V_p]^{2M}\ \forall z \in \overline{\Omega} \cap V_p. \tag{2.2}$$

Here, $2M$ represents the type of $\partial\Omega$ along $\boldsymbol{C}$. The above result is strongly reminiscent of [12, Lemma 2.1] by Noell. In that lemma, if $\boldsymbol{C}$ – where $\boldsymbol{C}$ is *not necessarily* complex-tangential, but $\partial\Omega$ is of type $2M$ along $\boldsymbol{C}$ – has the property that at each $p \in \boldsymbol{C}$ there is a holomorphic function, smooth upto

$\partial\Omega$, that peaks on a small closed sub-arc of $\boldsymbol{C}$ passing through p, then we can find a holomorphic peak function, smooth upto $\partial\Omega$, that satisfies the estimate (2.2). In our situation we do not, of course, have holomorphic functions that peak locally along $\boldsymbol{C}$. However, we can use some of Noell's ideas (which in turn rely on an estimate by Bloom [3]) and exploit the complex-tangency of $\boldsymbol{C}$ to construct an *almost-holomorphic* local peak function that satisfies good estimates. This construction is presented in Section 4.

We complete the proof of our main theorem in Section 5. Theorem 1.1-(i) follows from the construction of the family $\{h_k\}_{k\in\mathbb{N}}$ described above. Each h_k is, near $\boldsymbol{C}$, a holomorphic correction of the k-th power of G (G as introduced above). This correction is achieved by solving an appropriate $\overline{\partial}$-equation in $\overline{\Omega}$, and the estimate (2.2) is used to show that h_k satisfies the three properties listed above. Theorem 1.1-(ii) follows from the fact that in the real-analytic setting $\partial\Omega$ is of constant type along $\boldsymbol{C}$ *except* for a discrete set of points in $\boldsymbol{C}$. Using Theorem 1.1-(i) and the fact that each point in this discrete set is a peak point for $A(\Omega)$, we deduce Theorem 1.1-(ii).

3. A technical lemma

In this section, we present an abstract lemma that is instrumental to the proof of our main theorem. We begin with a definition :

Definition 3.1. Given an open set $V \subset \mathbb{R}^N$, a **bump function** f in V is a function belonging to $\mathcal{C}_c^\infty(V;[0,1])$ such that $\mathrm{int}[f^{-1}\{1\}] \neq \emptyset$.

Our technical lemma is as follows :

Lemma 3.2. *Let Ω be a bounded domain in $\mathbb{C}^2$ having smooth boundary and let $\boldsymbol{C}$ be a smooth curve in $\partial\Omega$. Assume that for each $p \in \boldsymbol{C}$, there exists a small neighbourhood V_p of p such that for each bump function $\chi \in \mathcal{C}_c^\infty(V_p;[0,1])$, for which $\mathrm{int}[\chi^{-1}\{1\}] \cap \boldsymbol{C}$ is an arc, we can find a sequence of functions $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ (depending on χ) satisfying*

(i) *$\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ is uniformly bounded on $\overline{\Omega}$.*
(ii) *$\lim_{k\to\infty} h_k(z) = 0$ $\forall z \in \overline{\Omega} \setminus (\boldsymbol{C} \cap V_p)$.*
(iii) *$\lim_{k\to\infty} h_k(z) = \chi(z)$ $\forall z \in \boldsymbol{C} \cap V_p$.*

Then, $\boldsymbol{C}$ is a countable union of peak-interpolation sets for $A(\Omega)$.

Remark 3.3. Before we prove the above lemma, we remark that a form of this lemma is true if Ω is a bounded domain in $\mathbb{C}^n$ and $\boldsymbol{C}$ is replaced by $\boldsymbol{M} \subset \partial\Omega$ – where $\boldsymbol{M}$ is a smooth submanifold of $\partial\Omega \cap U$, U being an open subset of $\mathbb{C}^n$. However, in order to be able to derive the conclusion of the above lemma in this new setting with $\dim_{\mathbb{R}}(\boldsymbol{M}) > 1$, one would have to produce, for *each* bump function $\chi \in \mathcal{C}_c^\infty(V_p;[0,1])$ (i.e. not *merely* for those χ for which $\mathrm{int}[\chi^{-1}\{1\}] \cap \boldsymbol{M}$ is nice), an $h \in A(\Omega)$ that would satisfy conditions (i)-(iii) above. Being able to find such an h could be rather difficult if $\dim_{\mathbb{R}}(\boldsymbol{M}) > 1$, because $\mathrm{int}[\chi^{-1}\{1\}] \cap \boldsymbol{M}$ could be structurally quite complicated in this situation. We add that if $\partial\Omega$ is *strictly* pseudoconvex, a less exacting form of the above lemma – see, for instance, [9, Lemma 6] – suffices to infer peak-interpolation in higher dimensions.

Proof. Fix $p \in \boldsymbol{C}$. We may assume that $\boldsymbol{C} \cap V_p$ is an arc in $\boldsymbol{C}$. Let K be any compact subset of $\boldsymbol{C} \cap V_p$ and let μ be any annihilating measure. Then

$$K = (\boldsymbol{C} \cap V_p) \ \setminus \ \coprod_{k \in \mathbb{N}} \mathcal{A}_k,$$

where each $\mathcal{A}_k$ is an *open* sub-arc of $\boldsymbol{C} \cap V_p$. If we could show that $\mu(\mathcal{A}_k) = 0$ for each k, and that $\mu(\boldsymbol{C} \cap V_p) = 0$ then, by the additivity of μ, we could conclude that $\mu(K) = 0$.

Let $\mathcal{C} \subset \boldsymbol{C} \cap V_p$ be any *closed* sub-arc of $\boldsymbol{C}$. Let $\{\mathfrak{D}_\nu\}_{\nu \in \mathbb{N}}$ be a shrinking family of compact subsets of $\mathbb{C}^2$ such that

(a) $\mathfrak{D}_{\nu+1} \subset \text{int}(\mathfrak{D}_\nu)$,
(b) $\cap_{\nu \in \mathbb{N}} \mathfrak{D}_\nu = \mathcal{C}$,
(c) $\mathfrak{D}_\nu \subseteq V_p$,
(d) $\boldsymbol{C} \cap \mathfrak{D}_\nu$ is an arc.

Let $\chi_\nu \in \mathcal{C}^\infty_c(V_p;\ [0,1])$ be a bump function with

$$\chi_\nu|_{\mathfrak{D}_{\nu+1}} \equiv 1, \qquad \text{supp}\ \chi_\nu \subseteq \mathfrak{D}_\nu.$$

Finally, define $\{h_{k,\nu}\}_{k \in \mathbb{N}}$ to be the sequence of functions corresponding to χ_ν given by the hypothesis of this lemma.

Choose any $\mu \perp A(\Omega)$. By the bounded convergence theorem

$$0 = \lim_{k \to \infty} \int_{\overline{\Omega}} h_{k,\nu} d\mu = \int_{\boldsymbol{C} \cap V_p} \chi_\nu d\mu.$$

Another passage to the limit yields $\mu(\mathcal{C}) = 0$, and this is true for any $\mu \perp A(\Omega)$. As μ is a regular measure, this shows that $\mu(\mathcal{A}) = 0$ for any *open* sub-arc $\mathcal{A} \subset \boldsymbol{C} \cap V_p$; in particular $\mu(\boldsymbol{C} \cap V_p) = 0$. Let $\mathcal{V}_p$ be any neighbourhood of p such that $\mathcal{V}_p \Subset V_p$. In view of our remarks in the first paragraph of this proof we have just shown that for any $\mu \perp A(\Omega)$, $|\mu|(\boldsymbol{C} \cap \overline{\mathcal{V}_p}) = 0$. By Bishop's theorem $\boldsymbol{C} \cap \overline{\mathcal{V}_p}$ is a peak-interpolation set for $A(\Omega)$. Letting p vary over a countable dense subset of $\boldsymbol{C}$, we have the desired result. $\square$

4. Constructing an almost holomorphic function that peaks locally on $\boldsymbol{C}$

Let $p \in \partial\Omega$. In this section, we will study $\partial\Omega$ near p with respect to a convenient system of local coordinates that are almost holomorphic with respect to $\boldsymbol{C}$ (near p) – where Ω and $\boldsymbol{C}$ are as in Theorem 1.1-(i). We first prove the following lemma which asserts the existence of local coordinates having the desired properties.

Lemma 4.1. *Let Ω be a bounded domain in $\mathbb{C}^2$ having smooth boundary and let $\boldsymbol{C} \subset \partial\Omega$ be a complex-tangential curve. Let $p \in \boldsymbol{C}$. There is a neighbourhood $\omega \ni p$ and a $\mathcal{C}^\infty$-diffeomorphism $\Phi : (\omega, p) \to (\mathbb{C}^2, 0)$ which is almost holomorphic with respect to $(\boldsymbol{C} \cap \omega)$ and so that, writing $(\zeta_1, \zeta_2) := \Phi(z_1, z_2)$, we have*

(1) $\Phi(\boldsymbol{C} \cap \omega) \subset \{(\zeta_1, \zeta_2) : \mathsf{im}(\zeta_1) = \zeta_2 = 0\}$.

(2) $\Phi(\partial\Omega \cap \omega)$ *is defined by a defining function of the form*

$$\rho(\zeta) = A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) - u_2,$$

where $\zeta_k := u_k + iv_k,\ k = 1,2$*;* $A(\zeta_1) = O(|\zeta_1|^2),\ R(\zeta_1, v_2) = O(|v_2|^2)$*, and*

$$A(u_1) = B(u_1) = 0,$$
$$\nabla A(u_1) = \nabla B(u_1) = 0,$$

for every u_1 *near* 0.

Proof. Without loss of generality, we may let p be the origin, and assume that, near p, $\partial\Omega$ is defined by

$$r(z_1, z_2) = h(z_1, \mathfrak{im}(z_2)) - \mathrm{Re}(z_2),$$

where $h(0) = 0$ and $\nabla h(0) = 0$.

Let ω be a neighbourhood of $p = 0$ and let $\mathcal{M} \subset \partial\Omega$ be the smooth 2-manifold of ω formed by the integral curves to the vector-field $-\mathbb{J}(\nabla r)$ passing through $(\boldsymbol{C} \cap \omega)$. $\mathcal{M}$ is totally real. Let $\gamma = (\gamma_1, \gamma_2) : (B(0;\varepsilon), (u_1, u_2) = 0) \to ((\mathcal{M} \cap \omega), p = 0)$ parametrize $\mathcal{M}$ near $p = 0$ in such a way that $\mathrm{Image}(\gamma|_{\{u_2=\mathrm{const}\}})$ are all integral curves to the unit section of $T(\mathcal{M}) \cap H(\partial\Omega)|_{\mathcal{M}}$, such that $\mathrm{Image}(\gamma|_{\{u_2=0\}}) = (\boldsymbol{C} \cap \omega)$ and such that $\frac{\partial\gamma(0,0)}{\partial u_2} = -\mathbb{J}(\nabla r)(0,0)$. Shrinking ω if necessary, we construct a diffeomorphism $\Phi : (\omega, p = 0) \to (\mathbb{C}^2, 0)$ of class $\mathcal{C}^\infty$, which is almost holomorphic with respect to $(\mathcal{M} \cap \omega)$, by defining

$$\Phi^{-1}(\zeta_1, \zeta_2) = (\Gamma_1(\zeta_1, -i\zeta_2),\ \Gamma_2(\zeta_1, -i\zeta_2)) := \eta(\zeta_1, \zeta_2),$$

where $\zeta_k := u_k + iv_k,\ k = 1,2$; and Γ_k is an almost holomorphic extension of $\gamma_k,\ k = 1,2$. Notice that by construction

$$\begin{aligned} \Phi(\mathcal{M} \cap \omega) &\subset \{(\zeta_1, \zeta_2) : v_1 = u_2 = 0\}, \\ \Phi(\boldsymbol{C} \cap \omega) &\subset \{(\zeta_1, \zeta_2) : v_1 = \zeta_2 = 0\}. \end{aligned} \tag{4.1}$$

Now, $\Phi(\partial\Omega \cap \omega)$ is defined by

$$\rho(\zeta_1, \zeta_2) = r \circ \Phi^{-1}(\zeta_1, \zeta_2).$$

We expand ρ around the origin in a Taylor series. We make use of the fact that Γ_k are almost holomorphic with respect to $\{(\zeta_1, \zeta_2)|\ v_1 = v_2 = 0\}$ to get

$$\rho(\zeta) = \ 2\ \mathrm{Re}\left[\sum_{j=1}^{2} \frac{\partial r}{\partial z_j}(\eta(0,0)) \left\{ \frac{\partial \Gamma_j}{\partial \zeta_1}(0,0)\zeta_1 + (-i)\frac{\partial \Gamma_j}{\partial \zeta_2}(0,0)\zeta_2 \right\}\right] + O(|\zeta|^2).$$

Using the fact that

$$\frac{\partial \Gamma_j}{\partial \zeta_k}(0,0) = \frac{\partial \gamma_j}{\partial u_k}(0,0), \quad j,k = 1,2$$

$$\begin{aligned}\rho(\zeta) =&\ 2\ \mathrm{Re}\left[\sum_{j=1}^{2}\frac{\partial r}{\partial z_j}(\gamma(0,0))\frac{\partial \gamma_j}{\partial u_1}(0,0)\zeta_1 + (-i)\sum_{j=1}^{2}\frac{\partial r}{\partial z_j}(\gamma(0,0))\frac{\partial \gamma_j}{\partial u_2}(0,0)\zeta_2\right] + O(|\zeta|^2)\\ =&\ 2\ \mathrm{Re}\left[(-i)\sum_{j=1}^{2}\frac{\partial r}{\partial z_j}(\gamma(0,0))\frac{\partial \gamma_j}{\partial u_2}(0,0)\zeta_2\right] + O(|\zeta|^2)\\ =& -u_2 + O(|\zeta|^2).\end{aligned}$$

The second equality follows from the complex-tangency of Image$[\gamma(\bullet,0)]$ which implies

$$\sum_{j=1}^{2}\frac{\partial r}{\partial z_j}(\gamma(u_1,0))\frac{\partial \gamma_j}{\partial u_1}(u_1,0) = 0 \quad \forall u_1 \in (-\varepsilon,\varepsilon),$$

and the last equality follows from the normalization condition on $\frac{\partial\gamma(0,0)}{\partial u_2}$. We see that the only term in the above expansion of first-order in either ζ_1 or ζ_2 is $-u_2$. Hence, the hypersurface $\Phi(\partial\Omega\cap\omega)$ is tangent at 0 to the hyperplane $\{(\zeta_1,\zeta_2)\in\mathbb{C}^2|u_2=0\}$ (in the remainder of this section, we will refer to this hyperplane as $\boldsymbol{H}$). Thus, we can find, near $0\in\mathbb{C}^2$, a defining function – and for convenience of notation, we will continue to call it ρ – which has the form

$$\rho(\zeta) = A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) - u_2, \tag{4.2}$$

where $A(\zeta_1) = O(|\zeta_1|^2)$ and $R(\zeta_1,v_2) = O(|v_2|^2)$. Then, since $\Phi(\boldsymbol{C}\cap\omega)\subset\Phi(\partial\Omega\cap\omega)$, setting $v_1=\zeta_2=0$ in (4.2), we get

$$A(u_1) = 0 \quad \forall (u_1,0)\in\Phi(\boldsymbol{C}\cap\omega). \tag{4.3}$$

And since $\Phi(\mathcal{M}\cap\omega)\subset\Phi(\partial\Omega\cap\omega)$, setting $v_1=u_2=0$ in (4.2), we see that $B(u_1)v_2+O(|v_2|^2)=0$ $\forall(u_1,v_2)$ belonging to a small neighbourhood 0. Thus

$$B(u_1) = 0 \quad \forall (u_1,0)\in\Phi(\boldsymbol{C}\cap\omega). \tag{4.4}$$

By construction, $(\nabla\rho)(u_1,v_2)$ is a normal vector to $\Phi(\mathcal{M}\cap\omega)$ $\forall(u_1,v_2)\in\Phi(\mathcal{M}\cap\omega)$. This implies that $T_{(u_1,v_2)}[\Phi(\partial\Omega\cap\omega)] = \boldsymbol{H}$ $\forall(u_1,v_2)\in\Phi(\mathcal{M}\cap\omega)$. Computing $(\nabla\rho)(u_1,v_2)$, we see that $\nabla A(u_1)+\nabla B(u_1)v_2=0$ $\forall(u_1,v_2)$ belonging to a neighbourhood of 0. Thus

$$\nabla A(u_1) = \nabla B(u_1) = 0 \quad \forall (u_1,0)\in\Phi(\boldsymbol{C}\cap\omega). \tag{4.5}$$

By (4.3), (4.4) and (4.5), we have the desired result. $\square$

We now state the key lemma of this paper. It concerns the construction of an almost holomorphic peak function of the type discussed in Section 2.

Proposition 4.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ of finite type, and let $\partial\Omega$ be of class $\mathcal{C}^\infty$. Let $\boldsymbol{C}\subset\partial\Omega$ be a complex-tangential curve of class $\mathcal{C}^\infty$, and let $\partial\Omega$ be of constant type $2M$ along $\boldsymbol{C}$. Let $p\in\boldsymbol{C}$. There exists a neighbourhood $V\equiv V(p)$ of p and a uniform constant $C>0$, and for any open set $U\Subset V$ such that $\boldsymbol{C}\cap U$ is an arc, there is a neighbourhood $V_1\equiv V(p,U)$ of p satisfying $\boldsymbol{C}\cap V_1=\boldsymbol{C}\cap V$ and a function $G\in\mathcal{C}^\infty(V_1)$ – G depending on p and U – which satisfies*

(1) $G^{-1}\{1\} = \boldsymbol{C} \cap \overline{U}$.
(2) $\overline{\partial} G$ *vanishes to infinite order on* $V \cap \boldsymbol{C}$.
(3) $|G(z)| \leq 1 - C\,dist[z, \boldsymbol{C} \cap V]^{2M}$ *for each* $z \in \overline{\Omega} \cap V_1$.

Proof. Let $\omega \ni p$ and $\Phi : (\omega, p) \to (\mathbb{C}^2, 0)$ be the change of coordinate described in Lemma 4.1. Let $\Phi(\partial\Omega \cap \omega)$ be defined by

$$\rho(\zeta_1, \zeta_2) = A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) - u_2. \tag{4.6}$$

Consider a point $(x_0, 0) \in \Phi(\boldsymbol{C} \cap \omega)$ and let

$$\varrho_{x_0}(\zeta_1^*, \zeta_2) = \mathcal{A}_{x_0}(\zeta_1^*) + \mathcal{B}_{x_0}(\zeta_1^*)v_2 + \mathcal{R}_{x_0}(\zeta_1^*, v_2) - u_2 \tag{4.7}$$

represent the expansion of ρ in (4.6) around $(x_0, 0)$, where $\zeta_1^* := \zeta_1 - x_0$.

Claim 1. Shrinking ω if necessary, there is a $c > 0$ such that

$$A(u_1 + iv_1) \geq cv_1^{2M}, \ \forall\zeta_1 \text{ such that } \zeta \in \Phi(\omega). \tag{4.8}$$

As $A(x_0) = B(x_0) = 0$ and $\nabla A(x_0) = \nabla B(x_0) = 0$ for each $(x_0, 0) \in \Phi(\partial\Omega \cap \omega)$, the right-hand side of (4.7) represents a defining function of the form (2.1). By Remark 2.2(3), the function $\mathcal{A}_{x_0}$ in (4.7) must vanish to order $2M$ at 0, whereby the function A in (4.6) must vanish precisely to order $2M$ at each $(u_1, 0) \in \Phi(\partial\Omega \cap \omega)$. Now write

$$A(u_1 + iv_1) = a_J(u_1)v_1^J + O(|v_1|^{J+1}), \tag{4.9}$$

where J is the least positive integer k such that $a_k \not\equiv 0$ near $u_1 = 0$. By our above remarks, it is clear that $J \leq 2M$. But, if $J < 2M$, then if $\widetilde{u_1}$ is such that $a_J(\widetilde{u_1}) \neq 0$, then A vanishes to order $< 2M$ at $u_1 + iv_1 = \widetilde{u_1}$, which contradicts our remarks above. Thus, $J = 2M$ in (4.9) and

$$A(u_1 + iv_1) = a_{2M}(u_1)v_1^{2M} + O(|v_1|^{2M+1}).$$

and $a_{2M}(0) \neq 0$. Now recall that Φ is almost-holomorphic with respect to $(\mathcal{M} \cap \omega)$. If, in fact $(u_1 + iv_1, u_2 + iv_2)$ were *holomorphic* coordinates, then the pseudoconvexity of Ω would have implied that

$$\alpha : (u_1, v_1) \mapsto a_{2M}(u_1)v_1^{2M} \text{ is subharmonic,}$$
$$\Delta\alpha(u_1, v_1) > 0 \text{ off } \{v_1 = 0\}, \text{ and } (u_1, v_1) \text{ close to } 0.$$

This would have implied that $a_{2M}(u_1) > 0$ for u_1 close to 0 (the second statement above follows from an obvious calculation). In our present situation, the coordinates $(u_1 + iv_1, u_2 + iv_2)$ differ from holomorphic ones by terms vanishing to arbitrarily high order along $(\boldsymbol{C} \cap \omega)$. From the last two facts, we can conclude that shrinking ω if necessary

$$a_{2M}(u_1) > 0, \quad \forall(u_1, 0) \in \Phi(\partial\Omega \cap \omega).$$

From this final fact, we deduce (4.8). Hence the claim.

Claim 2. We can find a $\omega_1 \Subset \omega$ and a uniform constant $T > 0$ such that

$$B(\zeta_1)^2 \leq TA(\zeta_1), \ \forall\zeta \in \Phi(\overline{\Omega} \cap \omega_1). \tag{4.10}$$

To see this, we use a procedure originating in [3, Secn. 3]. Write $q = (x_0, 0) \in \varPhi(\boldsymbol{C} \cap \omega)$. The positivity of the Levi-form for $\partial\Omega$ on the complex tangent vectors implies that *were* (u_1+iv_1, u_2+iv_2) *holomorphic coordinates* there would be a $\delta > 0$ such that the function $\mathfrak{L}$ induced by the Levi-form $\mathfrak{L} : D(x_0; \delta) \times (-\delta, \delta) \to \mathbb{R}$ defined by

$$\mathfrak{L} = |\partial_{\bar{2}}\rho|^2 \ \partial^2_{1\bar{1}}\rho + |\partial_{\bar{1}}\rho|^2 \ \partial^2_{2\bar{2}}\rho - 2\mathrm{Re}[\partial_1\rho \ \partial_{\bar{2}}\rho \ \partial^2_{\bar{1}2}\rho]$$

would be non-negative (notice that $\mathfrak{L}$ is independent of u_2). In our present situation, however, $\mathfrak{L}(u_1, v_2) \geq 0 \ \forall (u_1, v_2) \in \varPhi(\mathcal{M} \cap \omega)$.

Write

$$\mathfrak{L}(\zeta_1, v_2) = \mathfrak{L}^{(0)}(\zeta_1) + v_2\mathfrak{L}^{(1)}(\zeta_1) + v_2^2\mathfrak{L}^{(2)}(\zeta_1) + O(|v_2|^3).$$

It has been shown in [3] that if $\mathrm{ord}(B) < \mathrm{ord}(A)$, then

$$\begin{aligned} \mathrm{In}(\mathfrak{L}^{(0)}) &= \frac{1}{4}\mathrm{In}(\partial^2_{1\bar{1}}A), \qquad \mathrm{ord}(\mathfrak{L}^{(0)}) = \mathrm{ord}(A) - 2, \\ \mathrm{In}(\mathfrak{L}^{(1)}) &= \frac{1}{4}\mathrm{In}(\partial^2_{1\bar{1}}B), \qquad \mathrm{ord}(\mathfrak{L}^{(1)}) = \mathrm{ord}(B) - 2. \end{aligned} \tag{4.11}$$

If already $2\mathrm{ord}(B) \geq \mathrm{ord}(A)$, then (4.10) would follow trivially. Thus, *assume* that $2\mathrm{ord}(B) < \mathrm{ord}(A)$. Write $r = \mathrm{ord}(B)$. We have

$$\frac{1}{\lambda^{2r-2}}\mathfrak{L}(\lambda(u_1 + i0), \lambda^r v_2) \geq 0, \quad \forall (u_1, v_2) \in (x_0 - \delta, x_0 + \delta) \times (-\delta, \delta), \ \forall \lambda \in \mathbb{R}_+.$$

But from (4.11) and our assumption

$$\lim_{\lambda \to 0_+} \frac{1}{\lambda^{2r-2}}\mathfrak{L}(\lambda u_1, \lambda^r v_2) = \frac{v_2}{4}\mathrm{In}(\partial^2_{1\bar{1}}B)(\zeta_1). \tag{4.12}$$

Write

$$B(u_1 + iv_1) = b_J(u_1)v_1^J + O(|v_1|^{J+1}),$$

where J is the least positive integer k such that $b_k \not\equiv 0$ near $u_1 = 0$. By Lemma 4.1(2), $J \geq 2$, whence $\mathrm{In}(B)$ is non-harmonic near 0. So, as v_2 occurs linearly in the right-hand side of (4.12), it is impossible that

$$\frac{v_2}{4}\mathrm{In}(\partial^2_{1\bar{1}}B)(u_1) \geq 0, \quad \forall (u_1, v_2) \in (x_0 - \delta, x_0 + \delta) \times (-\delta, \delta).$$

This results in a contradiction. So $2\mathrm{ord}(B) \geq \mathrm{ord}(A)$, which, in conjunction with the positivity of A, viz. (4.8), yields (4.10).

Finally, define $H : \varPhi(\overline{\Omega} \cap \omega_1) \to \mathbb{C}$ by

$$H(\zeta) = \zeta_2 - \alpha\zeta_2^2 \ ,$$

for $\alpha > 0$ chosen appropriately large. We choose α as follows : Observe that

$$\begin{aligned} &\frac{1}{2}A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) + \frac{\alpha}{6}v_2^2 \\ &\qquad = \left(\frac{T}{\sqrt{2}}v_2 + \frac{B(\zeta_1)}{\sqrt{2T}}\right)^2 + \frac{1}{2T}[TA(\zeta_1) - B(\zeta_1)^2] - \frac{T^2}{2}v_2^2 + R(\zeta_1, v_2) + \frac{\alpha}{6}v_2^2. \end{aligned}$$

The first two terms of the right-hand side of the above equation are positive, in view of (4.10). So, we shrink ω_1 appropriately, and choose $\alpha > 0$ so large that

$$\frac{1}{2}A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) + \frac{\alpha}{6}v_2^2 \geq 0, \quad \forall \zeta \in \Phi(\overline{\Omega} \cap \omega_1). \tag{4.13}$$

Now consider :

Case (i). $u_2 \geq 0$. Let $\varepsilon_1 > 0$ be so small that $(u_2 - \alpha u_2^2) \geq u_2/2$ for $\zeta \in \Phi(\overline{\Omega} \cap B(p; \varepsilon_1))$ and such that $B(p; \varepsilon_1) \subset \omega_1$. Then, for all such ζ, we have :

$$\begin{aligned} \operatorname{Re}[H(\zeta)] = &\ (u_2 - \alpha u_2^2) + \alpha v_2^2 \\ \geq &\ \frac{1}{2}u_2 + \alpha v_2^2 \\ = &\ \frac{1}{4}u_2 + \frac{\alpha}{2}v_2^2 + \frac{1}{4}(u_2 + 2\alpha v_2^2) \\ \geq &\ \frac{1}{4}u_2 + \frac{\alpha}{2}v_2^2 + \frac{1}{4}\left[\{A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2)\} + 2\alpha v_2^2\right] \\ = &\ \frac{1}{4}u_2 + \frac{1}{8}A(\zeta_1) + \frac{\alpha}{2}v_2^2 + \frac{1}{4}\left[\frac{1}{2}A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) + 2\alpha v_2^2\right] \\ \gtrsim &\ u_2^2 + v_2^2 + A(\zeta_1) \quad [\text{using } (4.13)]. \end{aligned} \tag{4.14}$$

Case (ii). $u_2 < 0$. Let $\varepsilon_2 > 0$ be so small that $(u_2 - \alpha u_2^2) \geq 2u_2$ for $\zeta \in \Phi(\overline{\Omega} \cap B(p; \varepsilon_2))$ and such that $B(p; \varepsilon_2) \subset \omega_1$. Then, for all such ζ, we have (we argue exactly as before)

$$\begin{aligned} \operatorname{Re}[H(\zeta)] \geq &\ -u_2 + \frac{\alpha}{2}v_2^2 + 3\left(u_2 + \frac{\alpha}{6}v_2^2\right) \\ \geq &\ -u_2 + \frac{3}{2}A(\zeta_1) + \frac{\alpha}{2}v_2^2 + 3\left[\frac{1}{2}A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) + \frac{\alpha}{6}v_2^2\right] \\ \gtrsim &\ u_2^2 + v_2^2 + A(\zeta_1) \quad [\text{using } (4.13)]. \end{aligned} \tag{4.15}$$

Now let $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. From (4.8), (4.14) and (4.15) we see that there is a uniform constant $\kappa > 0$ such that

$$\begin{aligned} \operatorname{Re}[H(\zeta)] \geq &\ \kappa(u_2^2 + v_2^2 + v_1^{2M}) \\ \geq &\ \kappa \operatorname{dist}[\zeta, \Phi(\boldsymbol{C} \cap B(p; \varepsilon_0))]^{2M} \quad \forall \zeta \in \Phi(\overline{\Omega} \cap B(p; \varepsilon_0)). \end{aligned} \tag{4.16}$$

Write $\Phi(\boldsymbol{C} \cap U) = (a, b)$, and without loss of generality, we may assume that $a < 0 < b$. Define the function ϕ as follows

$$\phi(u_1) = \begin{cases} \exp\{1/(u_1 - a)\}, & \text{if } u_1 < a \\ 0, & \text{if } a \leq u_1 \leq b \\ \exp\{-1/(u_1 - b)\}, & \text{if } u_1 > b. \end{cases}$$

Let $r > 0$ such that $B(0; r) \supset \Phi[B(p; \varepsilon_0)]$, and let $R(\sigma)$ be the rectangle

$$R(\sigma) = \{(u_1 + iv_1) \in \mathbb{C} \mid |u_1| < r,\ |v_1| < \sigma\}.$$

By an argument given in Noell [12, Lemma 2.1], there exists a smooth almost holomorphic extension $\widetilde{\phi}$ of ϕ and a $\sigma > 0$ sufficiently small such that

$$\text{Re}[\widetilde{\phi}(u_1 + iv_1)] \geq -\frac{\kappa}{2} v_1^{2M}, \quad u_1 + iv_1 \in R(\sigma). \tag{4.17}$$

We set

$$V_1(p, U) = B(p; \varepsilon_0) \cap \Phi^{-1}[\text{Image}(\Phi) \cap (R(\sigma) \times \mathbb{C})].$$

From (4.16) and (4.17), we infer that the function $G(z) = (1-\widetilde{\phi}) \circ \Phi(z) - H \circ \Phi(z)$ satisfies (1)-(3). $\square$

5. The proof of Theorem 1.1

The proof of Theorem 1.1-(i) : Let $\boldsymbol{C}$ be as in Theorem 1.1-(i), and fix $p \in \boldsymbol{C}$. Let $V(p)$ be the neighbourhood of p as given by Proposition 4.2. We will use Lemma 3.2 to provide a proof. Take V_p, in the notation of that lemma, to be $V(p)$. In the notation of Lemma 3.2, let $\chi \in \mathcal{C}_c^\infty(V_p; [0,1])$ be a bump function such that $\text{int}[\chi^{-1}\{1\}] \cap \boldsymbol{C}$ is an arc. Write $U = \text{int}[\chi^{-1}\{1\}]$. Now let $V_1 = V_1(p, U)$ and $G \in \mathcal{C}^\infty(V_1)$ be as given by Proposition 4.2.

Define

$$G_k(z) = \begin{cases} [G(z)]^k \ \chi(z), & \text{if } z \in \overline{\Omega} \cap V_1 \\ 0, & \text{if } z \in \overline{\Omega} \setminus V_1. \end{cases}$$

Also define

$$f_k(z) = \overline{\partial} G_k(z) = k[G(z)]^{k-1} \ \overline{\partial} G(z) \ \chi(z) + [G(z)]^k \ \overline{\partial} \chi(z). \tag{5.1}$$

For a $(0,1)$ form $\phi(z) = \phi_1(z_1) d\overline{z}_1 + \phi_2(z_2) d\overline{z}_2$ defined on $\overline{\Omega}$, define

$$\|\phi\|_{\overline{\Omega}} := \max\{\sup_{\overline{\Omega}} |\phi_1(z)|, \ \sup_{\overline{\Omega}} |\phi_2(z)|\}.$$

By construction

$$\|G^k \ \overline{\partial} \chi\|_{\overline{\Omega}} \longrightarrow 0 \text{ as } k \to \infty. \tag{5.2}$$

Notice that $\overline{\partial} G$ vanishes to infinite order wherever $G(z) = 1$. Thus, for $j = 1, 2$

$$|k[G(z)]^{k-1} \ \partial_{\overline{j}} G(z) \ \chi_(z)| \lesssim k[1 - C\text{dist}[z, \boldsymbol{C} \cap V_p]^{2M}]^{k-1} |\partial_{\overline{j}} G(z)| \longrightarrow 0 \text{ uniformly as } k \to \infty. \tag{5.3}$$

From (5.2) and (5.3)

$$\|f_k\|_{\overline{\Omega}} \longrightarrow 0 \text{ as } k \to \infty. \tag{5.4}$$

Now consider the following $\overline{\partial}-$equations on Ω

$$\overline{\partial} u_k = f_k.$$

We need Lipschitz estimates for the solution of the $\overline{\partial}$-equation on pseudoconvex domains in $\mathbb{C}^2$ of finite type. Such estimates may be found in several places in the literature; for instance, in the results of Chang, Nagel & Stein [5], which imply that

$$\|u_k\|_{\overline{\Omega}} \ \leq \|u_k\|_{\Lambda^{1/N}(\overline{\Omega})} \ \leq C^* \|f_k\|_{\overline{\Omega}} \ , \tag{5.5}$$

where N is a positive integer such that $\tau(p) \leq N$ for each $p \in \partial\Omega$, $\Lambda^{1/N}(\overline{\Omega})$ is the class of complex-valued Lipschitz functions on $\overline{\Omega}$ of order $1/N$, and $C^* > 0$ is a constant depending only on Ω. From (5.4) and (5.5) we see that $\|u_k\|_{\overline{\Omega}} \to 0$, whence, defining

$$h_k(z) = G_k(z) - u_k(z), \quad \forall z \in \overline{\Omega}$$

we have a sequence of $A(\Omega)$ functions with

$$\lim_{k\to\infty} h_k(z) = \lim_{k\to\infty} G_k(z) = \begin{cases} \chi(z), & \text{if } z \in \boldsymbol{C} \cap V_p \\ 0, & \text{if } z \in \overline{\Omega} \setminus (\boldsymbol{C} \cap V_p). \end{cases}$$

Notice that, by construction, the sequence $\{h_k\}_{k\in\mathbb{N}}$ is uniformly bounded. $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ satisfies properties (i)–(iii) in the hypothesis of Lemma 3.2 for the bump function $\chi \in \mathcal{C}_c^\infty(V_p;[0,1])$ such that $\mathrm{int}[\chi^{-1}\{1\}] \cap \boldsymbol{C}$ is an arc. Thus we conclude, using Lemma 3.2, that any compact subset of $\boldsymbol{C}$ is a peak-interpolation set for $A(\Omega)$.

The proof of Theorem 1.1-(ii) : In the present situation, Ω is a bounded domain having a real-analytic boundary and $\boldsymbol{C}$ is a real-analytic complex-tangential curve. Let B be an open ball in $\mathbb{C}^2$ and let $\gamma : (-2\varepsilon, 2\varepsilon) \to \boldsymbol{C}$ be an injective real-analytic parametrization of $\boldsymbol{C}$ locally such that $\mathrm{Image}(\gamma|_{[-\varepsilon,\varepsilon]}) = (\boldsymbol{C} \cap \overline{B})$. Let $p \in (\boldsymbol{C} \cap \overline{B})$ be such that

$$\tau(p) = \min_{q\in\boldsymbol{C}\cap\overline{B}} \tau(q).$$

Write $\tau(p) = 2M$.

Recall that $H_p \otimes \mathbb{C}(\partial\Omega) = H_p^{1,0}(\partial\Omega) \oplus H_p^{0,1}(\partial\Omega)$, where $H \otimes \mathbb{C}(\partial\Omega)$ is the complexification of $H(\partial\Omega)$, and that $H_p^{1,0}(\partial\Omega)$ and $H_p^{0,1}(\partial\Omega)$ are the eigenspaces of the complex-structure map $\mathbb{J}$ corresponding to $+i$ and $-i$ respectively. Without loss of generality, we may assume that there is an open set $U \supset \overline{B}$ and a real-analytic section $\mathbf{L}$ of $H^{1,0}(\partial\Omega)|_U$ such that $\mathbf{L}(q)$ spans $H_q^{1,0}(\partial\Omega)$ and $\mathbf{L}(q) \in \{v \in H_q^{1,0}(\partial\Omega) : \|v\| = 1\}$ for each $q \in (\partial\Omega \cap U)$. Now consider the real-analytic function $\mathfrak{L} : S^1 \times I \to \mathbb{R}$ defined by

$$\mathfrak{L}(\zeta, t) = \sum_{\substack{j+k=2M \\ 1\leq j<2M}} \mathbf{L}^{j-1}\overline{\mathbf{L}}^{k-1}\langle[\mathbf{L}, \overline{\mathbf{L}}]\ ,\ \partial\rho\rangle(\gamma(t))\zeta^j\overline{\zeta}^k,$$

where I is an open interval around $[-\varepsilon, \varepsilon]$, S^1 is the unit circle in $\mathbb{C}$ and ρ is a defining function of $\partial\Omega$. Let t_0 be such that $\gamma(t_0) = p$. By the theorem of Bloom [4, Theorem 3.3], $\tau(p) = 2M$ implies that there exists a $\zeta_0 \in S^1$ such that $\mathfrak{L}(\zeta_0, t_0) \neq 0$. Then, by the real-analyticity of $\mathfrak{L}$, we conclude that

$$\{t \in [-\varepsilon, \varepsilon] : \mathfrak{L}(\zeta_0, t) = 0\} \text{ is a finite set } \mathfrak{S} \subset [-\varepsilon, \varepsilon].$$

Write $\mathfrak{S} = \{t_1, ..., t_N\}$. By [4, Theorem 3.3] again, in each connected component of $(\boldsymbol{C} \cap \overline{B}) \setminus \{\gamma_1(t_1), ..., \gamma(t_N)\}$, $\partial\Omega$ is of constant type $2M$. By Theorem 1.1-(i), therefore

$$(\boldsymbol{C} \cap \overline{B}) \setminus \{\gamma(t_1), ..., \gamma(t_N)\} \text{ is a countable union of peak-interpolation sets.} \tag{5.6}$$

Recall that Ω is a bounded domain with real-analytic boundary. By Bedford & Fornaess [1], therefore, every point of $\partial\Omega$ is a peak point for $A(\Omega)$. So, each $\gamma(t_j),\ j = 1, ..., N$; is a peak point for $A(\Omega)$.

This last fact, in conjunction with (5.6), implies that $\boldsymbol{C}$ is a countable union of peak-interpolation sets for $A(\Omega)$, and that each compact subset of $\boldsymbol{C}$ is a peak-interpolation set for $A(\Omega)$.

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