THE LOCAL POLYNOMIAL HULL NEAR A DEGENERATE CR SINGULARITY – BISHOP DISCS REVISITED

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Abstract. Let $\mathcal{S}$ be a smooth real surface in $\mathbb{C}^2$ and let $p \in \mathcal{S}$ be a point at which the tangent plane is a complex line. How does one determine whether or not $\mathcal{S}$ is locally polynomially convex at such a p — i.e. at a CR singularity ? Even when the order of contact of $T_p(\mathcal{S})$ with $\mathcal{S}$ at p equals 2, no clean characterisation exists; difficulties are posed by parabolic points. Hence, we study *non-parabolic* CR singularities. We show that the presence or absence of Bishop discs around certain non-parabolic CR singularities is completely determined by a Maslov-type index. This result subsumes all known facts about Bishop discs around order-two, non-parabolic CR singularities. Sufficient conditions for Bishop discs have earlier been investigated at CR singularities having high order of contact with $T_p(\mathcal{S})$. These results relied upon a subharmonicity condition, which fails in many simple cases. Hence, we look beyond potential theory and refine certain ideas going back to Bishop.

1. Introduction and statement of results

The simplest motive for this work is a rather naive one: we would like to know when, given a real surface $\mathcal{S} \subset \mathbb{C}^2$ and a point $p \in \mathcal{S}$ at which $T_p(\mathcal{S})$ is a complex line, $\mathcal{S}$ is locally polynomially convex at p. For $\mathcal{S}$ having only isolated exceptional points, this knowledge would enable one to determine whether $\mathcal{S}$ has a Stein neighbourhood basis. Insights into this naive problem would enable one to make tangible use of the many results about polynomial approximation on compact 2-submanifolds $\mathcal{S} \subset \mathbb{C}^2$ *with boundary*, most of which presuppose the polynomial convexity of $\mathcal{S}$.

We shall call a point of complex tangency a *CR singularity*. Consider a CR singularity $p \in \mathcal{S} \subset \mathbb{C}^2$ where the order of contact of $T_p(\mathcal{S})$ with $\mathcal{S}$ equals 2 — i.e. a *non-degenerate* CR singularity. Bishop showed [3] that there exist holomorphic coordinates (z, w) centered at p such that $\mathcal{S}$ is locally given (barring one manifestly locally polynomially convex case) by the equation $w = |z|^2 + \gamma(z^2 + \overline{z}^2) + G(z)$, where $\gamma \geq 0$, $G(z) = O(|z|^3)$, and three distinct situations arise. In Bishop's terminology, the CR singularity $p = (0, 0)$ is called elliptic if $0 \leq \gamma < 1/2$, parabolic if $\gamma = 1/2$, and hyperbolic if $\gamma > 1/2$. Bishop [3] showed that when $p \in \mathcal{S}$ is elliptic, the polynomially convex hull of $\mathcal{S}$ near p contains a one-parameter family of non-constant analytic discs attached to $\mathcal{S}$ that shrink to p. On the other hand, Forstnerič and Stout [7] showed that when p is hyperbolic, $\mathcal{S}$ is locally polynomially convex at p. In the applications hinted at, *we may not have the option of perturbing the given $\mathcal{S}$ at all*, whence the genericity of non-degenerate CR singularities cannot aid the study of such applications. Given this, one might ask what we can say about $(\mathcal{S}, p)$ if p is a *degenerate* CR singularity.

Even if p is a CR singularity in $\mathcal{S}$ where the order of contact of $T_p(\mathcal{S})$ with $\mathcal{S}$ equals 2, Jöricke's results in [8] show that the situation is far from tidy when p is a parabolic point.

2000 *Mathematics Subject Classification.* Primary 32E20, 46J10; Secondary 30E10.

Key words and phrases. Bishop disc, complex tangency, CR singularity, polynomially convex.

This work is supported by the DST via the Fast Track grant SR/FTP/MS-12/2007 and by the UGC under DSA-SAP, Phase IV.

One would expect some assumptions on the pair $(\mathcal{S}, p)$ (for p a degenerate CR singularity) for the outlines of a reasonable pattern, consistent with what is already known, to emerge. This motivates the following:

Definition 1.1. Let $\mathcal{S}$ be a $\mathcal{C}^k$-smooth real surface in $\mathbb{C}^2$, $k \geq 3$, and let $p \in \mathcal{S}$ be an isolated CR singularity. We say that p is *non-parabolic* if there exist an integer m, $2 \leq m < k$, and holomorphic coordinates (z, w) centered at p relative to which $\mathcal{S}$ has a local defining equation

$$\mathcal{S} \cap U_p : \quad w \ = \ \mathcal{F}_m(z, \overline{z}) + \mathcal{R}(z) \tag{1.1}$$

such that the graph $\Gamma(\mathcal{F}_m)$ has an isolated CR singularity at $(0,0) \in \mathbb{C}^2$. Here, $\mathcal{F}_m$ is a homogeneous polynomial in z and $\overline{z}$ of degree m and $\mathcal{R}$ is $O(|z|^{m+1})$.

Note that when $p \in \mathcal{S}$ is either elliptic or hyperbolic, it is non-parabolic in the sense of Definition 1.1. We wish to extend the Bishop/Forstnerič–Stout dichotomy (for non-parabolic, non-degenerate CR singularities) to the degenerate setting. When $(\mathcal{S}, p)$ is presented in the Bishop normal form near a non-parabolic, non-degenerate p, we have holomorphic coordinates (z, w) in which — using the notation of (1.1) — $\mathcal{F}_2$ is real-valued. This last fact is of central importance to Bishop's proofs in [3, Section 3]. This motivates the following:

Definition 1.2. Let $\mathcal{S}$ be a $\mathcal{C}^k$-smooth real surface in $\mathbb{C}^2$, $k \geq 3$, and let $p \in \mathcal{S}$ be an isolated CR singularity. Suppose $T_p(\mathcal{S})$ has finite order of contact $2 \leq m < k$ with $\mathcal{S}$ at p. We say that $\mathcal{S}$ *is thin at* p if there exist holomorphic coordinates (z, w) centered at p such that $\mathcal{S}$ is locally a graph of the form (1.1) above, and with respect to which $\mathcal{F}_m$ ($\mathcal{F}_m$ has the same meaning as in Definition 1.1) is real-valued.

When, for the pair $(\mathcal{S}, p)$, p is a non-parabolic, non-degenerate CR singularity (in which case $\mathcal{S}$ is *always* thin at p) the works [3] and [7], when read together, imply that the local polynomial convexity of $\mathcal{S}$ at p is determined precisely by the sign of a certain Maslov-type index, denoted by $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S}, p)$. Specifically:

($*$) *When $p \in \mathcal{S}$ is a non-parabolic, non-degenerate (hence thin) CR singularity, $\mathcal{S}$ is locally polynomially convex at p if and only if* $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S}, p) \leq 0$.

The reader is directed to Section 2 for the precise definition of the index $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S}, p)$. The goal of this paper is to attempt to extend ($*$) to non-parabolic, degenerate CR singularities. That brings us to our first theorem, which says, among other things, that $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S}, p) > 0 \Longrightarrow \mathcal{S}$ is *not* locally polynomially convex at p.

Theorem 1.3. *Let $\mathcal{S}$ be a $\mathcal{C}^k$-smooth real surface in $\mathbb{C}^2$, $k \geq 3$, and let $p \in \mathcal{S}$ be a CR singularity. Assume that p is non-parabolic and that $\mathcal{S}$ is thin at p. If* $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S}, p) > 0$*, then $\mathcal{S}$ is* not *locally polynomially convex at p.*

In fact, there exists a $\mathcal{C}^1$-smooth family of analytic discs whose boundaries are contained in $\mathcal{S}$. More precisely: there exist a neighbourhood $U_p \ni p$, an open interval $(0, R_0)$, and a function $\mathfrak{g} : (0, R_0) \longrightarrow A^\alpha(\mathbb{D}; \mathbb{C}^2)$ that is of class $\mathcal{C}^1$ on $(0, R_0)$ (for an arbitrary but fixed $\alpha \in (0,1)$), where each $\mathfrak{g}(r)$ is a non-constant analytic disc satisfying

i) $\mathfrak{g}(r)(\partial\mathbb{D}) \subset (\mathcal{S} \setminus \{p\}) \cap U_p \ \forall r \in (0, R_0)$*; and*
ii) $\mathfrak{g}(r)(\zeta) \longrightarrow \{p\}$ *for each* $\zeta \in \overline{\mathbb{D}}$ *as* $r \longrightarrow 0^+$.

Here, and elsewhere in this paper, $A^\alpha(\mathbb{D}; \mathbb{C}^2)$ denotes the class of all $\mathbb{C}^2$-valued maps on $\overline{\mathbb{D}}$ that are holomorphic on $\mathbb{D}$ and whose restrictions to $\partial\mathbb{D}$ are of Hölder class $\mathcal{C}^\alpha(\partial\mathbb{D})$.

Remark 1.4. The term "thin" must not be confused with the term "flat", which appears in the literature on polynomial convexity. $\mathcal{S}$ would be *flat* at p if $\mathcal{R}$ (as given by (1.1)) were *also* real-valued. The term "thin" arises in some parts of the literature on potential theory, but is unrelated to Definition 1.2.

Before commenting on the relationship between Theorem 1.3 and other results on the same theme in the literature, let us present a partial converse of Theorem 1.3.

Theorem 1.5. *Let $\mathcal{S}$ be a $\mathcal{C}^k$-smooth real surface in $\mathbb{C}^2$, $k \geq 3$, and let $p \in \mathcal{S}$ be a CR singularity. Assume that p is non-parabolic and that $\mathcal{S}$ is thin at p. Suppose* $\text{Ind}_{\mathcal{M}}(\mathcal{S}, p) \leq 0$.

1) *Let (z, w) be holomorphic coordinates centered at p such that (by hypothesis) $\mathcal{S}$ is locally defined by*

$$\mathcal{S} \cap U_p : \quad w \;=\; \mathfrak{F}_m(z, \overline{z}) + \mathcal{R}(z) \quad (\mathcal{R} \text{ is } O(|z|^{m+1}) \text{ for } |z| \text{ small}), \tag{1.2}$$

where $\mathfrak{F}_m$ is a real-valued polynomial that is homogeneous of degree m, $2 \leq m < k$. If $\mathcal{R}$ is real-valued, then $\mathcal{S}$ is locally polynomially convex at p.

2) *Suppose $k \geq 4$. Given any $\alpha \in (0, 1)$ (now $\mathcal{R}$ need not be real-valued), it is impossible to find a continuous one-parameter family $\mathfrak{g} : (0, 1) \longrightarrow A^{\alpha}(\mathbb{D}; \mathbb{C}^2)$ of immersed, non-constant analytic discs having all the following properties:*
   - $\mathfrak{g}(t)(\partial\mathbb{D}) \subset (\mathcal{S} \setminus \{p\}) \cap U_p \ \forall t \in (0, 1)$;
   - $\mathfrak{g}(t)(e^{i\cdot})$ *is a simple closed curve in* $\mathcal{S}$ $\forall t \in (0, 1)$; *and*
   - $\mathfrak{g}(t)(\zeta) \longrightarrow \{p\}$ *for each* $\zeta \in \overline{\mathbb{D}}$ *as* $t \longrightarrow 0^+$.

The point of Part (2) of Theorem 1.5 is to observe that, although we do not know whether $\text{Ind}_{\mathcal{M}}(\mathcal{S}, p) \leq 0$ implies that $\mathcal{S}$ is locally polynomially convex at p, the local polynomially convex hull of $(\mathcal{S}, p)$ does not contain non-constant analytic discs (with boundaries in $\mathcal{S} \setminus \{p\}$) that shrink to p. Note also that each part of Theorem 1.5 can be viewed as a partial converse to Theorem 1.3. These lead us to suggest the following conjecture:

Conjecture 1.6. *Let $\mathcal{S}$ be a $\mathcal{C}^k$-smooth real surface in $\mathbb{C}^2$, $k \geq 3$, and let $p \in \mathcal{S}$ be a CR singularity. Assume that p is non-parabolic and that $\mathcal{S}$ is thin at p. Then, $\mathcal{S}$ is locally polynomially convex at p if and only if* $\text{Ind}_{\mathcal{M}}(\mathcal{S}, p) \leq 0$.

The above conjecture may remind the reader of the findings of Jöricke [8] and Wiegerinck [14] on *parabolic* (non-degenerate) CR singularities. At least when $\text{Ind}_{\mathcal{M}}(\mathcal{S}, p) \neq 0$ parabolic points have been shown in [14, 8] to exhibit the conjectured dichotomy. The question arises as to why the ideas in [14, 8] should not reveal the same dichotomy when applied to non-parabolic, degenerate CR singularities. But the fact is, *without the condition of thinness, the dichotomy does not hold:* Wiegerinck has given examples [14, Section 4] in which $\text{Ind}_{\mathcal{M}}(\mathcal{S}, p) < 0$ and yet local polynomial convexity at p fails. The CR singularities studied by Wiegerinck are, for the most part, also non-parabolic, degenerate CR singularities. But, in place of thinness, they are required to satisfy a different analytical condition. The key point of departure of this article from [14] is summarised by these two observations:

- Although the surfaces $(\mathcal{S}, p)$ studied in [14] are not necessarily thin at p, *Wiegerinck's hypotheses do not hold true in general when $\mathcal{S}$ is thin at p.*
- Theorems 1.3 and 1.5 provide some evidence in support of Conjecture 1.6. In contrast, there does not seem to be a clear-cut discriminant for local polynomial convexity if thinness is replaced by the hypotheses in [14].

It is true that the class of pairs $(\mathcal{S}, p)$ with $\mathcal{S}$ being thin at p forms a small sub-case of the general situation. However, *this paper is devoted to studying a certain dichotomy.* Wiegerinck's examples suggest that very different considerations must apply when $(\mathcal{S}, p)$ is *not* thin. These considerations have been examined — although more from the viewpoint of detecting polynomial convexity than of polynomial hulls — in [1] and in a recent article [2].

As for the assumptions in Wiegerinck's work: we refer the reader to [14, Theorems 3.3, 3.4]. His assumptions, applied to our context, translate to the requirement that $\mathcal{F}_m$ must be subharmonic and non-harmonic. One of the motivations of this paper is to develop tools to show the existence of Bishop discs *in the absence of such subharmonicity conditions.* This is a meaningful motivation because of the following:

Fact (see Example 4.2)**.** *There exist polynomials $\mathcal{F}_m : \mathbb{C} \longrightarrow \mathbb{R}$, homogeneous of degree m, such that*

- 0 *is an isolated CR singularity of* $\Gamma(\mathcal{F}_m)$ *satisfying* $\text{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_m), 0) > 0$*; and*
- $\mathcal{F}_m$ *is* not *subharmonic.*

Example 4.2 rules out the possibility of simply applying the results of [14] to deduce Theorem 1.3.

Before proceeding to the proofs, we would like to point out a couple of new inputs required in the proof of Theorem 1.3, and to sketch the main ingredients of our approach. Our proof consists of the following parts:

- **Part I.** We work in the coordinate system (z, w) centered at p in which $(\mathcal{S}, p)$ is presented locally as shown in (1.2). We prove a general result:
$$\text{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_m), 0) \;=\; -\frac{\#[\mathcal{F}_m(e^{i\cdot})^{-1}\{0\} \cap [0, 2\pi)]}{2} + 1$$
(the notation $\#[S]$ stands for the cardinality of the set S). This tells us, since $\text{Ind}_{\mathcal{M}}(\mathcal{S}, p) > 0$, that we may assume (after making a holomorphic change of coordinate if necessary) that $\mathcal{F}_m(z) > 0 \;\forall z \in \mathbb{C} \setminus \{0\}$.
- **Part II.** We see that $\mathcal{F}_m^{-1}\{1\}$ is a simple closed real-analytic curve. Let g denote the boundary-value of the normalised Riemann mapping of $\mathbb{D}$ onto the region enclosed by $\mathcal{F}_m^{-1}\{1\}$. Then, the curves $\varphi_r : \partial\mathbb{D} \longrightarrow \mathbb{C}^2$, $r > 0$, given by $\zeta \longmapsto (rg(\zeta), r^m)$ are closed curves in $\Gamma(\mathcal{F}_m)$ that bound analytic discs. We view $\mathcal{S}$, equivalently the graph $\Gamma(\mathcal{F}_m + \mathcal{R})$, as a small perturbation of $\Gamma(\mathcal{F}_m)$, and attempt to obtain small corrections, say ψ_r, of φ_r $\forall r \in (0, R_0)$, for $R_0 > 0$ sufficiently small, such that $(\varphi_r + \psi_r)$ are curves in $\Gamma(\mathcal{F}_m + \mathcal{R})$ that bound analytic discs. This requirement gives us a family of functional equations, involving the harmonic-conjugate operator, parametrised by the interval $(0, R_0)$. The desired ψ_r, $r \in (0, R_0)$, are derived from the fixed points of these equations.
- **Part III.** One way to obtain fixed points is to show that the functionals involved in the aforementioned equations are contractions. This is the approach of Kenig & Webster in [9]. In making the required estimates, Kenig and Webster are aided by the following remarkable fact:
  - (▲) *If, in addition to the hypotheses in Theorem 1.3, the polynomial $\mathcal{F}_m$ is quadratic, then given any $l \in \mathbb{N}$, $l \geq 3$, there exists a holomorphic coordinate system (z, w) such that $(\mathcal{S}, p)$ has a local representation of the form* (1.2) *and such that* $\mathsf{Im}(\mathcal{R})(z) = O(|z|^{l+1})$*.*

  This fact is good enough to show that the Bishop discs foliate a $\mathcal{C}^\infty$-smooth 3-manifold with boundary. Unfortunately, *the conclusion of* (▲) *is not true in general if* $m > 2$. In the absence of (▲), we just make more stringent estimates.

These estimates turn out to be good enough to conclude that $(0, R_0) \ni r \longmapsto (\varphi_r + \psi_r)$ is $\mathcal{C}^1$-smooth.

One final expository remark is in order: one could set up a functional equation of the type that we allude to in Part II above, and naively hope to show that $r \longmapsto \psi_r$ is of class $\mathcal{C}^1$ using the Implicit Function Theorem. The problem is that, owing to the presence of the CR singularity, the relevant Fréchet (partial) derivative of the non-linear functional involved is *non-surjective at all the obvious zeros of this functional!* The reader's attention is drawn to the note in Step 2 of Section 3. It is this fact that leads to the (unavoidable) technicalities of the approach outlined above.

Since an important part of both Theorems 1.3 and 1.5 is based on a good understanding of $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S}, p)$, we shall begin with a discussion on this index in the next section. The proofs of Theorems 1.3 and 1.5 will be presented in Sections 3 and 5 respectively. A discussion on the non-subharmonicity of the local graphing functions of the $(\mathcal{S}, p)$ that we consider in this paper will be presented in Section 4.

2. Some facts about the Maslov-type index

Given a smooth real surface $\mathcal{S} \subset \mathbb{C}^2$, the term "Maslov-type index" might refer to three inter-related numbers that apply to slightly different contexts. They are:

a) *The index* $\mathrm{Ind}_{\mathcal{M},\gamma}(\mathcal{S})$ *of a closed path:* This applies to a closed path $\gamma : S^1 \longrightarrow \mathcal{S}$, where $\mathcal{S}$ is a totally-real submanifold of a region $\Omega \subseteq \mathbb{C}^2$.
b) *The index* $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S}, p)$ *of a CR singularity* p*:* This applies to a pair $(\mathcal{S}, p)$, where $\mathcal{S}$ is an orientable real 2-submanifold of some region $\Omega \subseteq \mathbb{C}^2$ having an isolated CR singularity at $p \in \mathcal{S}$.
c) *The index* $\mathrm{Ind}_{\mathcal{M},\psi}(\mathcal{S})$ *of an analytic disc* ψ*:* This applies to an analytic disc $\psi \in \mathcal{O}(\mathbb{D}; \mathbb{C}^2) \cap \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^2)$ with $\psi(\partial\mathbb{D}) \subset \mathcal{S}$, where $\mathcal{S}$ is a totally-real submanifold of some region $\Omega \subseteq \mathbb{C}^2$.

In this paper, it is the first two indices that will be relevant to our discussions. Before making the proper definitions, we will need one piece of notation. We set

$$G_{tot.\mathbb{R}}(\mathbb{C}^2) := \text{ the manifold of oriented totally-real planes in } \mathbb{C}^2,$$

where the differentiable structure on $G_{tot.\mathbb{R}}(\mathbb{C}^2)$ is the one that makes it a submanifold of the Grassmanian $G(2, \mathbb{R}^4)$ of oriented 2-subspaces of $\mathbb{R}^4$. We are now in a position to make our definitions. In doing so, we follow the constructions by Forstnerič in [6]. Here, we make one remark: we wish to define the concepts (a) and (b) above with the least amount of technicality possible, and to draw upon some computations in [6] that pertain to graphs in $\mathbb{C}^2$. Hence, in the definitions below *we will assume that the bundle* $\gamma^* \, T\mathcal{S}|_{\gamma(S^1)}$ *is a trivial bundle* (where $\gamma : S^1 \longrightarrow \mathcal{S}$ is as in (a)), although the notion of $\mathrm{Ind}_{\mathcal{M},\gamma}(\mathcal{S})$ is not restricted to the trivial-bundle case.

Definition 2.1. Let $\mathcal{S}$ be a totally-real 2-submanifold of a region $\Omega \subseteq \mathbb{C}^2$. Let $\gamma : S^1 \longrightarrow \mathcal{S}$ be a smooth, closed path such that the pullback $\gamma^* \, T\mathcal{S}|_{\gamma(S^1)}$ is a trivial bundle (equivalently, $\mathcal{S}$ is orientable along γ). Let $\Theta_\gamma : S^1 \longrightarrow G_{tot.\mathbb{R}}(\mathbb{C}^2)$ denote the tangent map, i.e. $\Theta_\gamma(\zeta) := T_{\gamma(\zeta)}(\mathcal{S})$. There is a well-defined map $\mathfrak{G} : G_{tot.\mathbb{R}}(\mathbb{C}^2) \longrightarrow \mathbb{C} \setminus \{0\}$ given by

$$\mathfrak{G}(P) := \det \left[X_1^P \;\; X_2^P \right], \;\; (X_1^P, X_2^P) \text{ a positively oriented orthonormal basis of } P,$$

this being well-defined because, given two positively oriented orthonormal bases (X_1^P, X_2^P) and (Y_1^P, Y_2^P), $Y_j^P = A(X_j^P)$, $j = 1, 2$, for some $A \in SL(2, \mathbb{R})$. The composition $\mathfrak{G} \circ \Theta_\gamma$

induces a homomorphism in homology $H_1(\mathfrak{G}\circ\Theta_\gamma) : H_1(S^1;\mathbb{Z}) \longrightarrow H_1(\mathbb{C}\setminus\{0\};\mathbb{Z})$. The degree of this homomorphism is called the *Maslov-type index of the path* γ, denoted by $\mathrm{Ind}_{\mathcal{M},\gamma}(\mathcal{S})$.

Definition 2.2. Let $\mathcal{S}$ be a real orientable 2-submanifold of some region $\Omega \subseteq \mathbb{C}^2$ that has an isolated CR singularity at $p \in \mathcal{S}$. Then there is an $\mathcal{S}$-open neighbourhood of p, say W_p, that is contractible to p and such that p is the only CR singularity in W_p. Let W_p have the orientation induced by the complex line $T_p(\mathcal{S})$. Let $\gamma : S^1 \longrightarrow W_p\setminus\{p\}$ be a smooth, simple closed curve that has positive orientation with respect to the orientation of W_p. Then, we define the *Maslov-type index of the CR singularity* p, written as $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S},p)$, by $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S},p) := \mathrm{Ind}_{\mathcal{M},\gamma}(W_p\setminus\{p\})$.

We note that $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S},p)$ is well-defined because $\mathrm{Ind}_{\mathcal{M},\gamma}(W_p\setminus\{p\})$ depends only on the homology class of γ in $W_p\setminus\{p\}$. When $\mathcal{S}$ is the graph $\Gamma(F)$ of some function F that is $\mathcal{C}^1$-smooth near $0\in\mathbb{C}$, with $\Gamma(F)$ having an isolated CR singularity at the origin, then $\gamma^*\, T\mathcal{S}|_{\gamma(S^1)}$ is trivial for any $\gamma : S^1 \longrightarrow \Gamma(F)\setminus\{(0,0)\}$ as in Definition 2.2. Using an explicit frame for $\gamma^*\, T\Gamma(F)|_{\gamma(S^1)}$, Forstnerič has shown that:

Lemma 2.3 ([6], Lemma 8)**.** *Let* Ω *be a domain in* $\mathbb{C}$ *containing* 0 *and let* $F\in\mathcal{C}^1(\Omega;\mathbb{C})$*. Suppose that the graph* $\Gamma(F)$ *has an isolated CR singularity at* 0*. Let* $\gamma : S^1 \longrightarrow \Omega\setminus\{0\}$ *be a smooth, positively-oriented, simple closed curve that encloses* 0 *and encloses no other points belonging to* $(\partial F/\partial\bar{z})^{-1}\{0\}$*. Then*

$$\mathrm{Ind}_{\mathcal{M}}(\Gamma(F),0) \;=\; \mathsf{Wind}\left(\frac{\partial F}{\partial \bar{z}}\circ\gamma, 0\right), \tag{2.1}$$

where the expression on the right-hand side denotes the winding number around 0*.*

We are now in a position to prove a key lemma. This was informally stated in Part I of our outline, in Section 1, of the proof of Theorem 1.3.

Lemma 2.4. *Let* $\mathfrak{F}_m : \mathbb{C} \longrightarrow \mathbb{R}$ *be a polynomial that is homogeneous of degree* m *and such that* $(\partial\mathfrak{F}_m/\partial\bar{z})^{-1}\{0\} = \{0\}$*. Then*

$$\mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathfrak{F}_m),0) \;=\; -\frac{\#[\mathfrak{F}_m(e^{i\cdot})^{-1}\{0\}\cap[0,2\pi)]}{2} + 1 \tag{2.2}$$

(the notation $\#[S]$ *denotes the cardinality of the set* S*).*

Proof. Let us define the real-analytic, 2π-periodic function f by the relation $\mathfrak{F}_m(z) = |z|^m f(\theta)$, where we write $z = |z|e^{i\theta}$. Then, we compute

$$\frac{\partial\mathfrak{F}_m}{\partial\bar{z}}(e^{i\theta}) \;=\; \frac{e^{i\theta}}{2}\left\{mf(\theta) + if'(\theta)\right\}. \tag{2.3}$$

We record two facts:

a) Since $f\in\mathcal{C}^\omega(\mathbb{R})$ and 2π-periodic, $\#\{\theta\in[0,2\pi) : f(\theta)=0\}$ is an even number.
b) Since $(\partial\mathfrak{F}_m/\partial\bar{z})^{-1}\{0\} = \{0\}$, $f(\theta)$ and $f'(\theta)$ cannot simultaneously vanish for any $\theta\in[0,2\pi)$.

Thus, we have two closed paths $\gamma_1,\gamma_2 : [0,2\pi] \longrightarrow \mathbb{C}\setminus\{0\}$, defined by:

$$\gamma_1(\theta) \;:=\; mf(\theta) + if'(\theta),$$
$$\gamma_2(\theta) \;:=\; e^{i\theta}.$$

Recalling that the winding number is additive across products, we get:

$$\mathsf{Wind}\left(\frac{\partial\mathfrak{F}_m}{\partial\bar{z}}(e^{i\cdot}), 0\right) \;=\; \mathsf{Wind}(\gamma_1,0) + \mathsf{Wind}(\gamma_2,0).$$

Hence, in view of the above and Lemma 2.3, it suffices for us to show that

$$\mathsf{Wind}(\gamma_1, 0) \;=\; -\frac{\#\{\theta \in [0, 2\pi) : f(\theta) = 0\}}{2}. \tag{2.4}$$

Let us first consider the case when $f^{-1}\{0\} \neq \emptyset$. Without loss of generality, we may assume that $f(0) = 0$. Let

$$0 = \theta_1 < \theta_1 < \dots \theta_N < 2\pi$$

denote the distinct zeros of $f|_{[0,2\pi)}$. Let $\phi : [0, 2\pi] \longrightarrow \mathbb{R}$ be a function having the following properties (recall that by (b) above f has only simple zeros):

- $\phi(\theta_j) = 0,\ j = 1, \dots, N$;
- $\phi'(\theta_j)f'(\theta_j) > 0,\ j = 1, \dots, N$;
- $|\phi'(\theta_j)| > |f'(\theta_j)|,\ j = 1, \dots, N$;
- $\left[\phi|_{(\theta_{j-1},\theta_j)}\right]'$ has *precisely* one simple zero in $(\theta_{j-1}, \theta_j),\ j = 1, \dots, N$; and
- ϕ has a $\mathcal{C}^\infty$-smooth periodic extension to $\mathbb{R}$.

In view of the third property of ϕ, there exists a constant $K > 0$ such that

$$|\phi'(\theta_j)| \;>\; |f'(\theta_j)| \;>\; K,\ j = 1, \dots, N. \tag{2.5}$$

Define the homotopy $H : [0, 2\pi] \times [0, 1] \longrightarrow \mathbb{C}$ by

$$H(\theta, t) \;:=\; m[(1-t)f(\theta) + t\phi(\theta)] + i[(1-t)f'(\theta) + t\phi'(\theta)].$$

Note that, by construction

$$\mathsf{Re}(H)(\theta, t) = 0 \iff f(\theta) = 0,$$
$$f(\theta) = 0 \implies |(1-t)f'(\theta) + t\phi'(\theta)| > K.$$

Hence, in fact, $H([0, 2\pi] \times [0, 1]) \subset \mathbb{C} \setminus \{0\}$. Thus γ_1 is homotopic in $\mathbb{C} \setminus \{0\}$ to the path $\mathit{\Gamma}_1 : [0, 2\pi] \longrightarrow \mathbb{C} \setminus \{0\}$ given by

$$\mathit{\Gamma}_1(\theta) \;=\; m\phi(\theta) + i\phi'(\theta),\ \ \theta \in [0, 2\pi].$$

By construction, the number of times that $\mathit{\Gamma}_1$ winds around the origin is half the number of times that $\mathit{\Gamma}_1$ intersects the real axis. But, since, by construction, $\mathit{\Gamma}_1$ is oriented clockwise, we get, by homotopy invariance of the winding number:

$$\mathsf{Wind}(\gamma_1, 0) \;=\; \mathsf{Wind}(\mathit{\Gamma}_1, 0) \;=\; -\frac{\#\{\theta \in [0, 2\pi) : f(\theta) = 0\}}{2}. \tag{2.6}$$

In the case when $f^{-1}\{0\} = \emptyset$, γ_1 never crosses the real axis. Hence

$$f^{-1}\{0\} = \emptyset \implies \mathsf{Wind}(\gamma_1, 0) \;=\; 0. \tag{2.7}$$

From (2.6) and (2.7) we see that (2.4) has been established. This establishes our result. □

The last result in this section provides a Maslov-index calculation for the graph of a homogeneous polynomial $\mathfrak{F}_m$ that is, in contrast to Lemma 2.4, *complex*-valued. It will find no application later in this paper, but we present it as it might be of independent interest.

Lemma 2.5. *Let $\mathfrak{F}_m$ be a non-holomorphic, complex-valued polynomial that is homogeneous of degree m and such that $(\partial\mathfrak{F}_m/\partial\overline{z})^{-1}\{0\} = \{0\}$. Define the polynomial $\mathcal{Q}_m \in \mathbb{C}[z, w]$ by the relation*

$$\mathcal{Q}_m(z, \overline{z}) \;=\; \frac{\partial\mathfrak{F}_m}{\partial\overline{z}}(z, \overline{z})$$

by making explicit the dependence of $\partial\mathfrak{F}_m/\partial\overline{z}$ on z and $\overline{z}$. Let $\mathfrak{p}_m$ be the polynomial defined as $\mathfrak{p}_m(z) := \mathcal{Q}_m(z,1)$. Then,

$$\mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathfrak{F}_m),0) \;=\; 2\left(\sum\left\{\mu(\zeta) : \zeta \in \mathfrak{p}_m^{-1}\{0\}\bigcap\mathbb{D}\right\}\right) - (m-1),$$

where $\mu(\zeta)$ denotes the multiplicity of ζ as a zero of the polynomial $\mathfrak{p}_m$.

Proof. Note that, by hypothesis, the path $(\partial\mathfrak{F}_m/\partial\overline{z})(e^{i\cdot})$ does not pass through the origin. Hence, in view of (2.1), we can explicitly compute the desired winding number to get:

$$\mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathfrak{F}_m),0) \;=\; \frac{1}{2\pi i}\int_0^{2\pi} \frac{\partial^2_{z\overline{z}}\mathfrak{F}_m(e^{i\theta})ie^{i\theta} - \partial^2_{\overline{z}\overline{z}}\mathfrak{F}_m(e^{i\theta})ie^{-i\theta}}{\partial_{\overline{z}}\mathfrak{F}_m(e^{i\theta})}d\theta. \tag{2.8}$$

We now compute that

$$\partial^2_{z\overline{z}}\mathfrak{F}_m(e^{i\theta})ie^{i\theta} - \partial^2_{\overline{z}\overline{z}}\mathfrak{F}_m(e^{i\theta})ie^{-i\theta} \;=\; ie^{i\theta}\left[\frac{1}{z^{m-1}}\mathcal{Q}_m(z^2,1)\right]'_{z=e^{i\theta}}, \tag{2.9}$$

$$\partial_{\overline{z}}\mathfrak{F}_m(e^{i\theta}) \;=\; \frac{1}{z^{m-1}}\mathcal{Q}_m(z^2,1)\Big|_{z=e^{i\theta}}. \tag{2.10}$$

From (2.8), (2.9) and (2.10), we get

$$\mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathfrak{F}_m),0) \;=\; \frac{1}{2\pi i}\oint_{S^1}\frac{\left[\frac{1}{z^{m-1}}\mathfrak{p}_m(z^2)\right]'_{z=\zeta}}{\frac{1}{\zeta^{m-1}}\mathfrak{p}_m(\zeta^2)}d\zeta.$$

Since, by hypothesis, the denominator in the above integral never vanishes, the Argument Principle gives us

$$\mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathfrak{F}_m),0) \;=\; 2\left(\sum\left\{\mu(\zeta) : \zeta \in \mathfrak{p}_m^{-1}\{0\}\bigcap\mathbb{D}\right\}\right) - (m-1).$$

□

3. The proof of Theorem 1.3

We introduce some notations that will be needed in the proof of Theorem 1.3. First, we define the Banach space $\mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{F})$, $\alpha \in (0,1)$, where $\mathbb{F}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$ in the following proof, as

$$\mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{F}) := \left\{f : \partial\mathbb{D} \longrightarrow \mathbb{F} : \sup_{\theta\in\mathbb{R}}|f(e^{i\theta})| + \sup_{\theta\neq\phi\in\mathbb{R}}\frac{|f(e^{i\theta}) - f(e^{i\phi})|}{|\theta-\phi|^\alpha} < \infty\right\},$$

where the norm on this Banach space is:

$$\|f\|_{\mathcal{C}^\alpha} \;:=\; \sup_{\theta\in\mathbb{R}}|f(e^{i\theta})| + \sup_{\theta\neq\phi\in\mathbb{R}}\frac{|f(e^{i\theta}) - f(e^{i\phi})|}{|\theta-\phi|^\alpha}.$$

We will also have occasion to use the following abbreviation

$$[f]_\alpha \;:=\; \sup_{\theta\neq\phi\in\mathbb{R}}\frac{|f(e^{i\theta}) - f(e^{i\phi})|}{|\theta-\phi|^\alpha}.$$

In what follows, $A(\partial\mathbb{D})$ will denote the class of restrictions to the unit circle of functions that are holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. For any $f \in \mathcal{C}(\partial\mathbb{D};\mathbb{F})$ we will denote the Fourier series of f as follows:

$$f \;\sim\; \sum_{n\in\mathbb{Z}}\widehat{f}(n)e^{in\theta}.$$

It is well known that if $f \in \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{F})$ with $\alpha \in (0,1)$, then any harmonic conjugate on $\mathbb{D}$ of the Poisson integral of f extends to a function on $\overline{\mathbb{D}}$ and its restriction to $\partial\mathbb{D}$, say h_f, is of class $\mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{F})$. In this paper, $\mathfrak{H}[f]$ will denote that h_f which satisfies (in our Fourier-series notation) $\widehat{h_f}(0) = 0$. In terms of Fourier series:

$$\mathfrak{H}[f] \sim \sum_{n\in\mathbb{Z}} -i \ \mathrm{sgn}(n)\widehat{f}(n)e^{in\theta}.$$

We call $\mathfrak{H}[f]$ the *conjugate of* f. Recall that the operator $\mathfrak{H} : \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{F}) \longrightarrow \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{F})$ is a certain singular-integral operator that is bounded on $\mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{F})$. We shall use this fact (which we assume the reader is familiar with) in Step 1 of our proof below.

The proof of Theorem 1.3. Let $(\mathcal{S}, p)$ be as stated in the hypothesis of the theorem. By definition, there is a neighbourhood U_p of p and holomorphic coordinates (z,w) centered at p such that $\mathcal{S}$ is locally defined by

$$\mathcal{S} \cap U_p : \quad w \ = \ \mathcal{F}_m(z) + \mathcal{R}(z) \quad (\text{for } |z| \text{ small}), \tag{3.1}$$

where $\mathcal{F}_m$ is a real-valued polynomial that is homogeneous of degree m, and $\mathcal{R}(z) = O(|z|^{m+1})$. From this last fact, and from (2.1) in Lemma 2.3, we see that

$$\mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_m + \mathcal{R}), 0) \ = \ \mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_m), 0).$$

This is seen by considering the relevant winding numbers of small circles centered at $z = 0$. Now note that the index $\mathrm{Ind}_{\mathcal{M}}(\mathcal{S}, p)$ is, by construction, invariant under holomorphic changes of coordinate. Hence

$$\mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_m), 0) \ = \ \mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_m + \mathcal{R}), 0) \ = \ \mathrm{Ind}_{\mathcal{M}}(\mathcal{S}, p) \ > \ 0.$$

Applying (2.2) to the above statement, we may conclude, without loss of generality, that

$$\mathcal{F}_m(z) \ > \ 0 \ \ \forall z \in \mathbb{C} \setminus \{0\}. \tag{3.2}$$

Let us define

$$\rho \ := \ \sup\{s > 0 : \Gamma(\mathcal{F}_m + \mathcal{R}; \overline{D(0;s)}) \subset U_p, \text{ and } \overline{D(0;s)} \times \{0\} \subset U_p\},$$

(here, and elsewhere in this paper, $\Gamma(\mathcal{F}_m + \mathcal{R}; \overline{D(0;s)})$ denotes the portion of the graph of $(\mathcal{F}_m + \mathcal{R})$ over the closed disc $\overline{D(0;s)}$). In the remainder of this proof, *whenever we use the parameter* $r > 0$, *we will assume that* $0 < r < 3\rho/4$. In view of (3.2), and the fact that $\mathcal{F}_m$ is homogeneous, $\mathcal{F}_m^{-1}\{1\}$ is a real-analytic curve that meets each ray originating at 0 at precisely one point. To see this, we first note that, by homogeneity, for each fixed $\theta \in [0, 2\pi)$, $\mathcal{F}_m(re^{i\theta}) = C_\theta r^m \ \forall r > 0$. By (3.2), $C_\theta > 0$, whence the ray $\{re^{i\theta} : r > 0\}$ intersects $\mathcal{F}_m^{-1}\{1\}$ precisely at $e^{i\theta}/C_\theta^{1/m}$. It now follows from basic topology that $\mathcal{F}_m^{-1}\{1\}$ is a simple closed curve that encloses 0. Thus, we can define

$$\begin{aligned} \mathcal{D} \ &:= \ \text{the region in } \mathbb{C} \text{ enclosed by } \mathcal{F}_m^{-1}\{1\}, \\ G \ &:= \ \text{the unique Riemann mapping of } \mathbb{D} \text{ onto } \mathcal{D} \text{ such that } G(0) = 0,\ G'(0) > 0. \end{aligned}$$

Note that as $\mathcal{F}_m^{-1}\{1\}$ is a real-analytic, simple closed curve:

- G extends to a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{\mathcal{D}}$ such that $G : (\mathbb{D}, \partial\mathbb{D}) \longrightarrow (\mathcal{D}, \partial\mathcal{D})$; and
- By the Schwarz Reflection Principle, $\exists \varepsilon > 0$ such that G extends to a function $\widetilde{G} \in \mathcal{O}(D(0; 1+\varepsilon))$ such that $\widetilde{G}'(\zeta) \neq 0 \ \forall \zeta \in \partial\mathbb{D}$.

Let us define $g := \widetilde{G}\Big|_{\partial\mathbb{D}}$ and $g_r := \kappa r g$, where $\kappa := (1/2)\left[\sup_{\zeta\in\partial\mathbb{D}}|g(\zeta)|\right]^{-1}$.

Step 1. *Constructing the relevant Bishop's Equation*

Let us fix an $\alpha \in (0,1)$. Define the mapping $\mathcal{A} : \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R}) \longrightarrow \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{C}) \cap A(\partial\mathbb{D})$ by

$$\mathcal{A}[\psi] \ := \ \psi + i\mathfrak{H}[\psi].$$

Recall that $\mathfrak{H}[\psi]$ denotes the conjugate of ψ. It is well-known that for each $\alpha \in (0,1)$, there exists a $\gamma_\alpha > 0$ such that

$$\|\mathfrak{H}[\psi]\|_{\mathcal{C}^\alpha} \ \leq \ \gamma_\alpha \|\psi\|_{\mathcal{C}^\alpha} \ \ \forall \psi \in \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R}). \tag{3.3}$$

We remark here that the optimal dependence of γ_α on the parameter $\alpha \in (0,1)$ is known; see [10, Part IV, Theorem 2.4]. Specifically, if ψ is in the Hölder class $\mathcal{C}^{k,\alpha}(\partial\mathbb{D};\mathbb{F})$ then

$$\|\mathfrak{H}[\psi]\|_{\mathcal{C}^{k,\alpha}} \ \leq \ C(k)\tfrac{1}{\alpha(1-\alpha)}\|\psi\|_{\mathcal{C}^{k,\alpha}},$$

where $C(k) > 0$ denotes a constant that depends only on k. However, we shall not require this degree of precision in the arguments that follow and we shall work with $\gamma_\alpha > 0$. Define the open set $\Omega_\alpha \subset \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R})$ by

$$\Omega_\alpha \ := \ \{\psi \in \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R}) : \sqrt{1+\gamma_\alpha^2}\,\|\psi\|_{\mathcal{C}^\alpha} < 3\rho/8\}.$$

Finally, define the function $\Phi : \Omega_\alpha \times (0, 3\rho/4) \longrightarrow \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R})$ by

$$\begin{aligned}\Phi(\psi, r) :=& -(\kappa r)^m + \mathfrak{F}_m \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi]) + (\mathsf{Re}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi]) \\ &+ \mathfrak{H}[(\mathsf{Im}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi])] \\ =&\partial_z\mathfrak{F}_m(g_r)e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi] + \partial_{\overline{z}}\mathfrak{F}_m(g_r)\overline{e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi]} + Q(g_r, e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi]) \\ &+ (\mathsf{Re}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi]) + \mathfrak{H}[(\mathsf{Im}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi])]\end{aligned} \tag{3.4}$$

where we define

$$Q(X,Y) \ := \ \sum_{j=2}^{m}\sum_{\mu+\nu=j}\frac{1}{\mu!\nu!}\partial_z^\mu\partial_{\overline{z}}^\nu\mathfrak{F}_m(X)Y^\mu\overline{Y}^\nu.$$

We are now in a position to assert the following:

Fact A. *If, for some $(\psi_0, r^0) \in \Omega_\alpha \times (0, 3\rho/4)$, $\Phi(\psi_0, r^0) = 0$, then there is an analytic disc $F \in \mathcal{O}(\mathbb{D};\mathbb{C}^2) \cap \mathcal{C}^\alpha(\overline{\mathbb{D}})$, which is a small perturbation of the analytic disc $(g_{r^0}, (\kappa r^0)^m)$, such that $F(\partial\mathbb{D}) \subset \mathcal{S}$.*

To justify the above assertion, note that as $\Phi(\psi_0, r^0)$ is identically zero,

$$\Phi(\psi_0, r^0) + i\mathcal{A}[(\mathsf{Im}\mathcal{R}) \circ (g_{r^0} + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_0])]$$

is the boundary value of a holomorphic function. However

$$\begin{aligned}\Phi(\psi_0, r^0) &+ i\mathcal{A}[(\mathsf{Im}\mathcal{R}) \circ (g_{r^0} + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_0])] \\ &= -(\kappa r^0)^m + \mathfrak{F}_m \circ (g_{r^0} + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_0]) + \mathcal{R} \circ (g_{r^0} + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_0]).\end{aligned} \tag{3.5}$$

Clearly, the Poisson integral of the function

$$(g_{r^0}, (\kappa r^0)^m) + \left(e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_0],\, i\mathcal{A}[(\mathsf{Im}\mathcal{R}) \circ (e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_0] + g_{r^0})]\right)$$

is an analytic disc $F := (F_1, F_2)$, and by (3.5)

$$F_2(\zeta) \ = \ \mathfrak{F}_m \circ F_1(\zeta) + \mathcal{R} \circ F_1(\zeta) \ \ \forall \zeta \in \partial\mathbb{D},$$

which is precisely the fact asserted above.

To show that the analytic discs described in Theorem 1.3 vary smoothly with respect to the parameter r, we have to establish that each of these discs exists. To this end, the above discussion helps in setting the following

Intermediate Goal. *To solve the equation $\Phi(\psi, r) = 0$ for all sufficiently small values of the parameter r.*

Step 2. *Setting up an equivalent equation to the functional equation $\Phi(\psi, r) = 0$*
Consider the linear operator (which is bounded from $\mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R})$ to $\mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R})$ owing to (3.3) above)

$$\begin{aligned}\Lambda_r : \psi \longmapsto\ & \partial_z \mathfrak{F}_m(g_r) e^{i\cdot}\mathcal{A}[\psi] + \partial_{\bar{z}} \mathfrak{F}_m(g_r) \overline{e^{i\cdot}\mathcal{A}[\psi]} \\ &= 2\mathsf{Re}\left\{\partial_z \mathfrak{F}_m(g_r) e^{i\cdot}\mathcal{A}[\psi]\right\}.\end{aligned}$$

Note. Before we engage in technicalities, we ought to point out the difficulties inherent in this problem. First note that:

The Fréchet (partial) derivative $\partial_\psi \Phi|_{(\psi,0)}$ is not invertible for any $\psi \in \Omega_\alpha$.

Suppose that could show that the Fréchet derivative $\partial_\psi \Phi|_{(\psi^0,r^0)}$ is invertible for some $(\psi^0, r^0) \in \Omega_\alpha \times (0, 3\rho/4) = \mathsf{Dom}(\Phi)$. With this, *we would still be unable to invoke the Implicit Function Theorem to either assert the existence of analytic discs attached to $\mathcal{S}$ or to infer their smooth dependence on r in a neighbourhood of r_0.* This is because it must first be established that $\Phi(\psi^0, r^0) = 0$! This is precisely our Intermediate Goal above. The inquisitive reader is directed also to Remark 3.1 below.

Claim. *Λ_r is an isomorphism.*
To show that Λ_r is surjective, note that it suffices to show that given any $f \in \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R})$, there exists a function $a_f \in A_0^\alpha(\partial\mathbb{D})$, where

$$A_0^\alpha(\partial\mathbb{D}) \ := \ \{h \in \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{C}) : \widehat{h}(0) \in \mathbb{R}, \text{ and } \widehat{h}(j) = 0 \ \forall j \leq -1\},$$

such that $2\mathsf{Re}\left\{\partial_z \mathfrak{F}_m(g_r) e^{i\cdot} a_f\right\} = f$. Note that, from the discussion preceding Step 2, it can be inferred that

$$\mathfrak{F}_m \circ (r\kappa\widetilde{G}) - (\kappa r)^m \ \textit{vanishes on } \partial\mathbb{D}.$$

Thus, recalling that $\widetilde{G}'(\zeta) \neq 0 \ \forall \zeta \in \partial\mathbb{D}$, there exists a $\delta > 0$ and a function $R \in \mathcal{C}^\omega(\mathrm{Ann}(0; 1-\delta, 1+\delta))$ such that (we treat r as a parameter here)

- $R(z) > 0 \ \forall z \in \mathrm{Ann}(0; 1-\delta, 1+\delta)$; and
- $\mathfrak{F}_m \circ (r\kappa\widetilde{G}) - (\kappa r)^m = r^m R(z)(|z|^2 - 1) \ \forall z \in \mathrm{Ann}(0; 1-\delta, 1+\delta)$.

By the chain rule (recall that g_r is the restriction of a holomorphic function):

$$e^{i\cdot}(\partial_z \mathfrak{F}_m) \circ (g_r) \ = \ \left.\frac{\partial(\mathfrak{F}_m \circ (\kappa r\widetilde{G}) - (\kappa r)^m)}{\partial z}\right|_{\partial\mathbb{D}} \times \frac{e^{i\cdot}}{\kappa r\widetilde{G}'} \tag{3.6}$$

$$= \ r^m \left(\bar{z}R|_{\partial\mathbb{D}}\right) \frac{e^{i\cdot}}{\kappa r\widetilde{G}'}. \tag{3.7}$$

The second equality follows from the fact that $(|z|^2 - 1)\partial_z R(z)$ vanishes on $\partial\mathbb{D}$. Hence, the desired a_f is a solution to the equation

$$2\frac{r^{m-1}R(e^{i\cdot})}{\kappa}\mathsf{Re}\left(\frac{a_f}{\widetilde{G}'}\right) \ = \ f \ \textit{ with } a_f \in A_0^\alpha(\partial\mathbb{D}). \tag{3.8}$$

It was shown by Privalov that – owing to the normalisation condition that a_f belong to $A_0^\alpha(\partial\mathbb{D})$ – the equation (3.8) has a *unique* solution in $A_0^\alpha(\partial\mathbb{D})$ given by

$$a_f(\zeta) \ = \ \frac{\kappa\widetilde{G}'(\zeta)}{2r^{m-1}} \,\mathcal{A}\left[\frac{f}{R(e^{i\cdot})}\right](\zeta) \ \ \forall \zeta \in \partial\mathbb{D}.$$

This establishes that Λ_r is surjective, and the uniqueness of a_f establishes that it is injective. Hence the claim.

To complete the discussion on the invertibility of Λ_r we note that $\Lambda_r^{-1} = \mathbb{A}_r$, where

$$\mathbb{A}_r[f] \;=\; \mathsf{Re}\left\{\frac{\kappa\widetilde{G}'(e^{i\cdot})}{2r^{m-1}}\,\mathcal{A}\left[\frac{f}{R(e^{i\cdot})}\right]\right\}.$$

Furthermore, from the fact that $\mathcal{A} = \mathbb{I}_{\mathcal{C}^\alpha} + i\mathfrak{H}$, and from the estimate (3.3), we get the following important estimate: there exists a $K_\alpha > 0$ such that

$$\|\mathbb{A}_r[f]\|_{\mathcal{C}^\alpha} \;\leq\; K_\alpha r^{1-m}\,\|f\|_{\mathcal{C}^\alpha} \;\; \forall f \in \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R}). \tag{3.9}$$

Finally, by applying $\mathbb{A}_r$ to the equation (3.4), we see that solving the equation

$$\Phi(\psi, r) \;=\; 0, \;\; (\psi, r) \in \Omega_\alpha \times (0, 3\rho/4)$$

is equivalent to solving

$$\begin{aligned}\psi + \mathbb{A}_r&\left[Q(g_r, e^{i\cdot}\mathcal{A}[\psi]) + (\mathsf{Re}\mathcal{R})\circ(e^{i\cdot}\mathcal{A}[\psi] + g_r) + \mathfrak{H}[(\mathsf{Im}\mathcal{R})\circ(e^{i\cdot}\mathcal{A}[\psi] + g_r)]\right]\\ &\equiv\; \psi - H(\psi; r)\\ &=\; 0.\end{aligned}$$

In view of this, the goal presented at the end of Step 1 is modified as follows:

Modified Intermediate Goal. *To find a fixed point of the map $H(\cdot; r) : \Omega_\alpha \longrightarrow \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R})$ for each sufficiently small value of the parameter r.*

Step 3. *Some estimates*
We shall use the contraction mapping principle to establish the modified goal above. For this purpose, we will (for a fixed $r > 0$) determine the image under $H(\cdot; r)$ of a small closed ball in $\mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R})$ centered at 0. We will also show that $H(\cdot; r)$ is a contraction on this ball. This requires some estimates.

Since $\mathcal{R}(z) = O(|z|^{m+1})$, it follows that there is a large positive constant $L > 0$ that is independent of $r > 0$ such that

$$\|(\mathsf{Re}\mathcal{R})\circ g_r\|_\infty \;\leq\; L\left(\frac{r}{\rho}\right)^{m+1}, \tag{3.10}$$

and

$$[(\mathsf{Re}\mathcal{R})\circ g_r]_\alpha \;\leq\; L\left(\frac{r}{\rho}\right)^{m+1}\|\mathsf{Re}\mathcal{R}\|_{\mathcal{C}^1}\kappa\rho \sup_{\theta\neq\phi\in\mathbb{R}}\frac{|g(e^{i\theta}) - g(e^{i\phi})|}{|\theta-\phi|^\alpha}. \tag{3.11}$$

We now set the stage for showing that for each $r > 0$ sufficiently small, $H(\cdot; r)$ is a contraction on the closed ball $\overline{\mathbb{B}_{\mathcal{C}^\alpha}(0; r^{1+\delta})}$, where we pick and fix $\delta \in (1/2, 1)$. Furthermore, we shall work with $r \in (0, r_1)$, where $r_1 > 0$ is so small that $r/(100\sqrt{1+\gamma_\alpha^2}) \geq r^{1+\delta}$ $\forall r \in (0, r_1)$. This will ensure that all values of ψ under consideration satisfy the constraint

$$\|\psi\|_{\mathcal{C}^\alpha} \;\leq\; \frac{r}{100\sqrt{1+\gamma_\alpha^2}}. \tag{3.12}$$

In the next few estimates, *we shall assume that these constraints are in effect even if not explicitly stated.* To simplify notation we set $A(\mu, \nu, \psi) := \mathcal{A}[\psi]^\mu\overline{\mathcal{A}[\psi]}^\nu$. We first estimate:

$$
\begin{aligned}
&\left\|Q(g_r, e^{i\cdot}\mathcal{A}[\psi_1]) - Q(g_r, e^{i\cdot}\mathcal{A}[\psi_2])\right\|_\infty \\
&\quad \leq \sum_{j=2}^{m}\sum_{\mu+\nu=j} \frac{r^{m-j}}{\mu!\nu!} \left\|\partial_z^\mu \partial_{\bar z}^\nu \mathfrak{F}_m(\kappa g)\left(A(\mu,\nu,\psi_1) - A(\mu,\nu,\psi_2)\right)\right\|_\infty \\
&\quad \leq \sum_{j=2}^{m}\sum_{\mu+\nu=j} \frac{r^{m-j}}{\mu!\nu!} \sup_{\zeta\in\overline{\mathbb{D}}} |\partial_z^\mu \partial_{\bar z}^\nu \mathfrak{F}_m(\zeta)| \\
&\qquad\qquad \times \sqrt{1+\gamma_\alpha^2}^{\,j-1} \left\|\mathcal{A}[\psi_1-\psi_2]\right\|_\infty \left(\mu r^{(1+\delta)(j-1)} + \nu r^{(1+\delta)(j-1)}\right) \\
&\quad \leq C_\alpha \sum_{j=2}^{m} j r^{(m-1)+\delta(j-1)} \left\|\psi_1-\psi_2\right\|_{\mathcal{C}^\alpha}. \qquad (3.13)
\end{aligned}
$$

It is the bound $\|\psi_j\|_{\mathcal{C}^\alpha} \leq r^{(1+\delta)}$, $j = 1, 2$, that leads to the second inequality above. Next, we estimate, using the fundamental theorem of calculus:

$$
\begin{aligned}
&\left\|(\mathsf{Re}\mathcal{R})\circ(g_r + e^{i\cdot}\mathcal{A}[\psi_1]) - (\mathsf{Re}\mathcal{R})\circ(g_r + e^{i\cdot}\mathcal{A}[\psi_2])\right\|_\infty \\
&\leq 2\sup_{\theta\in\mathbb{R}} \left|\mathsf{Re}\left\{\int_0^1 \partial_z(\mathsf{Re}\mathcal{R})(g_r(e^{i\theta}) + e^{i\theta}(t\mathcal{A}[\psi_1] + (1-t)\mathcal{A}[\psi_2])(e^{i\theta}))dt\right\}\right| \left\|\mathcal{A}[\psi_1-\psi_2]\right\|_\infty.
\end{aligned}
$$

Since $\mathcal{R}(z) = O(|z|^{m+1})$, and since the constraint (3.12) ensures that

$$
\mathrm{range}(g_r + te^{i\cdot}\mathcal{A}[\psi_1] + (1-t)e^{i\cdot}\mathcal{A}[\psi_2]) \;\Subset\; \mathsf{dil}_r[\mathrm{domain}(\mathcal{R})] \;\; \forall t\in[0,1]
$$

(where dil_r denotes the dilation on $\mathbb{C}$ by a factor of r), there is a large positive constant $L > 0$ that is independent of $r > 0$ such that

$$
\begin{aligned}
&\left\|(\mathsf{Re}\mathcal{R})\circ(g_r + e^{i\cdot}\mathcal{A}[\psi_1]) - (\mathsf{Re}\mathcal{R})\circ(g_r + e^{i\cdot}\mathcal{A}[\psi_2])\right\|_\infty \\
&\qquad\qquad \leq L\sqrt{1+\gamma_\alpha^2}\left\|\psi_1-\psi_2\right\|_{\mathcal{C}^\alpha}\left(\frac{r}{\rho}\right)^m. \qquad (3.14)
\end{aligned}
$$

To simplify the presentation of our next estimate, let us set

$$
C(j,\alpha) \;:=\; \left\|e^{ij\cdot}\right\|_{\mathcal{C}^\alpha}, \;\; j\in\mathbb{Z}, \quad M_\alpha \;:=\; \max\left(\|\mathcal{A}[\psi_1]\|_{\mathcal{C}^\alpha}, \|\mathcal{A}[\psi_2]\|_{\mathcal{C}^\alpha}\right),
$$

and write $j = \mu+\nu$, $2 \leq j \leq m$. Note that:

$$
\begin{aligned}
&\partial_z^\mu \partial_{\bar z}^\nu \mathfrak{F}_m(g_r)A(\mu,\nu,\psi_1) - \partial_z^\mu \partial_{\bar z}^\nu \mathfrak{F}_m(g_r)A(\mu,\nu,\psi_2) \\
&= r^{m-j}\partial_z^\mu \partial_{\bar z}^\nu \mathfrak{F}_m(\kappa g)e^{i(\mu-\nu)\cdot} \\
&\quad \times\Big(\overline{\mathcal{A}[\psi_1-\psi_2]}\sum_{t=0}^{\nu-1}\overline{\mathcal{A}[\psi_1]}^{t} A(\mu,\nu-t-1,\psi_2) + \mathcal{A}[\psi_1-\psi_2]\sum_{s=0}^{\mu-1} A(s,\nu,\psi_1)\mathcal{A}[\psi_2]^{\mu-s-1}\Big).
\end{aligned}
$$

Then, it is easy to see that

$$
\begin{aligned}
&\left[\partial_z^\mu \partial_{\bar z}^\nu \mathfrak{F}_m(g_r)A(\mu,\nu,\psi_1) - \partial_z^\mu \partial_{\bar z}^\nu \mathfrak{F}_m(g_r)A(\mu,\nu,\psi_2)\right]_\alpha \\
&\leq C_\alpha r^{m-j}\left\|\mathfrak{F}_m\right\|_{\mathcal{C}^{j+1}(\overline{\mathbb{D}})}\kappa\sup_{\theta\neq\phi\in\mathbb{R}}\frac{|g(e^{i\theta})-g(e^{i\phi})|}{|\theta-\phi|^\alpha} jM_\alpha^{j-1}\left\|\mathcal{A}[\psi_1-\psi_2]\right\|_\infty \\
&\quad + C_\alpha r^{m-j}\left\|\mathfrak{F}_m\right\|_{\mathcal{C}^{j}(\overline{\mathbb{D}})} jM_\alpha^{j-1}\left(\left\|\mathcal{A}[\psi_1-\psi_2]\right\|_\infty(C(\mu-\nu,\alpha)+(j-1)) + \left\|\mathcal{A}[\psi_1-\psi_2]\right\|_{\mathcal{C}^\alpha}\right)
\end{aligned}
$$

From this estimate, and the fact that $0 \leq M_\alpha \leq \sqrt{1+\gamma_\alpha^2}\, r^{1+\delta}$, we conclude that there exists a constant $C_\alpha > 0$, depending only on α, such that

$$\big[Q(g_r, e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_1]) - Q(g_r, e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_2])\big]_\alpha \;\leq\; C_\alpha \sum_{j=2}^{m} r^{(m-1)+\delta(j-1)} \|\psi_1 - \psi_2\|_{\mathcal{C}^\alpha}. \tag{3.15}$$

Finally, using exactly the same technique that led to the estimate (3.14), we compute:

$$\begin{aligned}
&\big[(\mathsf{Re}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_1]) - (\mathsf{Re}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_2])\big]_\alpha \\
&\leq\; 2\sup_{\theta\in\mathbb{R}} \left| \mathsf{Re}\left\{ \int_0^1 \partial_z(\mathsf{Re}\mathcal{R})(g_r(e^{i\theta}) + e^{i\theta}(t\mathcal{A}[\psi_1] + (1-t)\mathcal{A}[\psi_2])(e^{i\theta}))dt \right\} \right| \|\mathcal{A}[\psi_1 - \psi_2]\|_{\mathcal{C}^\alpha} \\
&\qquad + L\left(\frac{r}{\rho}\right)^m \|\mathcal{A}[\psi_1 - \psi_2]\|_\infty \int_0^1 \rho \left[\kappa g + \frac{e^{i\boldsymbol{\cdot}}(t\mathcal{A}[\psi_1] + (1-t)\mathcal{A}[\psi_2])}{r}\right]_\alpha dt.
\end{aligned}$$

Arguing in an analogous manner as above, we conclude that there exists a large uniform constant $L > 0$ and a $C_\alpha > 0$, depending only on α, such that:

$$\begin{aligned}
&\big[(\mathsf{Re}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_1]) - (\mathsf{Re}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_2])\big]_\alpha \\
&\qquad\qquad \leq\; 2L\left(\frac{r}{\rho}\right)^m \left(\sqrt{1+\gamma_\alpha^2} + C_\alpha r^\delta\right) \|\psi_1 - \psi_2\|_{\mathcal{C}^\alpha}.
\end{aligned} \tag{3.16}$$

We are now in a position to write down three key estimates that we need. In each of the three estimates, there exists a constant $C_\alpha > 0$ that depends only on α such that the following inequalities hold. Firstly, from (3.13) and (3.15) we get

$$\big\|Q(g_r, e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_1]) - Q(g_r, e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_2])\big\|_{\mathcal{C}^\alpha} \leq\; C_\alpha \sum_{j=2}^{m} r^{(m-1)+\delta(j-1)} \|\psi_1 - \psi_2\|_{\mathcal{C}^\alpha}. \tag{3.17}$$

Next, from (3.14) and (3.16), we get

$$\big\|(\mathsf{Re}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_1]) - (\mathsf{Re}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_2])\big\|_{\mathcal{C}^\alpha} \;\leq\; C_\alpha(1 + r^\delta) r^m \|\psi_1 - \psi_2\|_{\mathcal{C}^\alpha}. \tag{3.18}$$

Finally, note that the same arguments that lead to (3.14) and (3.16) also yield *exactly* analogous estimates for $(\mathsf{Im}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi])$. This observation, coupled with the bound (3.3) for the operator $\mathfrak{H}$ gives us

$$\begin{aligned}
&\big\|\mathfrak{H}\left[(\mathsf{Im}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_1]) - (\mathsf{Im}\mathcal{R}) \circ (g_r + e^{i\boldsymbol{\cdot}}\mathcal{A}[\psi_2])\right]\big\|_{\mathcal{C}^\alpha} \\
&\qquad\qquad \leq\; C_\alpha(1 + r^\delta) r^m \|\psi_1 - \psi_2\|_{\mathcal{C}^\alpha}.
\end{aligned} \tag{3.19}$$

All these estimates hold for $\psi_1, \psi_2 \in \overline{\mathbb{B}_{\mathcal{C}^\alpha}(0; r^{1+\delta})}$.

Step 4. *Completing the proof*
Applying the bounds (3.9) for the operator $\mathbb{A}_r$ to the estimates (3.17), (3.18) and (3.19), we see that there exists a constant $L_\alpha > 0$ such that

$$\begin{aligned}
\|H(\psi_1; r) - H(\psi_2; r)\|_{\mathcal{C}^\alpha} \;&\leq\; \frac{L_\alpha}{r^{m-1}} \left(r^m(1 + r^\delta) + \sum_{j=2}^{m} r^{(m-1)+\delta(j-1)} \right) \|\psi_1 - \psi_2\|_{\mathcal{C}^\alpha} \\
&\leq 2L_\alpha r^\delta \|\psi_1 - \psi_2\|_{\mathcal{C}^\alpha} \\
&\quad \forall \psi_1, \psi_2 \in \overline{\mathbb{B}_{\mathcal{C}^\alpha}(0; r^{1+\delta})} \ \textit{and} \ \forall r \in (0, r_2),
\end{aligned} \tag{3.20}$$

where $r_2 \in (0, r_1)$ is so small that the second inequality is valid for all $r \in (0, r_2)$. Furthermore, we deduce from the estimates (3.10) and (3.11) (and by the same argument that leads to the estimate (3.19)) that there exists a constant $K_\alpha > 0$ such that

$$\|H(0;r)\|_{\mathcal{C}^\alpha} \;=\; \|\mathbb{A}_r\left[(\mathsf{Re}\mathcal{R})\circ g_r + \mathfrak{H}[(\mathsf{Im}\mathcal{R})\circ g_r]\right]\|_{\mathcal{C}^\alpha} \;\leq\; K_\alpha r^2 \;\; \forall r \in (0, 3\rho/4). \tag{3.21}$$

Let $r_3 > 0$ be so small that:

$$\begin{aligned} 2L_\alpha r^\delta &\leq 1/2 \quad \textit{and} \\ K_\alpha r^2 &\leq r^{1+\delta}/2 \;\; \forall r \in (0, r_3). \end{aligned}$$

Set $R_0 := \min(3\rho/4, r_2, r_3)$. Then, for any $r \in (0, R_0)$,

$$\begin{aligned} &\psi_1, \psi_2 \in \overline{\mathbb{B}_{\mathcal{C}^\alpha}(0; r^{1+\delta})} \\ &\qquad \Longrightarrow \|H(\psi_1; r) - H(\psi_2; r)\|_{\mathcal{C}^\alpha} \leq (1/2)\, \|\psi_1 - \psi_2\|_{\mathcal{C}^\alpha} \quad [\textit{due to } (3.20)], \end{aligned} \tag{3.22}$$

and, furthermore

$$\begin{aligned} \psi \in \overline{\mathbb{B}_{\mathcal{C}^\alpha}(0; r^{1+\delta})} \implies \|H(\psi; r)\|_{\mathcal{C}^\alpha} &\leq (1/2)\, \|\psi\|_{\mathcal{C}^\alpha} + \|H(0;r)\|_{\mathcal{C}^\alpha} \quad && [\textit{due to } (3.22)] \\ &\leq r^{1+\delta}. && [\textit{due to } (3.21)] \end{aligned}$$

This last fact and the estimate (3.22) enable us to apply the contraction mapping principle to $H(\boldsymbol{\cdot}; r) : \overline{\mathbb{B}_{\mathcal{C}^\alpha}(0; r^{1+\delta})} \longrightarrow \overline{\mathbb{B}_{\mathcal{C}^\alpha}(0; r^{1+\delta})}$ for each $r \in (0, R_0)$ — owing to which we get:

Fact B. *For each $r \in (0, R_0)$, there exists a unique $\psi_r \in \overline{\mathbb{B}_{\mathcal{C}^\alpha}(0; r^{1+\delta})}$ such that $H(\psi_r; r) = \psi_r$.*

Before proceeding any further, we record that (shrinking $R_0 > 0$ further if necessary) $\|\psi_r\|_{\mathcal{C}^\alpha}$ *is not comparable to* $\|g_r\|_{\mathcal{C}^\alpha}$ $(\approx r)$ $\forall r \in (0, R_0)$, *which ensures that the desired analytic discs will be non-constant.* Let us now write $G(r) := \psi_r$. Recalling the discussions at the end of Step 1 and Step 2 of this proof, we see that the desired analytic discs $\mathfrak{g}(r)$ are the analytic maps defined by the boundary condition:

$$\mathfrak{g}(r)|_{\partial\mathbb{D}} \;=\; (g_r, (\kappa r)^m) + \left(e^{i\boldsymbol{\cdot}}\mathcal{A}[G(r)],\; i\mathcal{A}[(\mathsf{Im}\mathcal{R})\circ(e^{i\boldsymbol{\cdot}}\mathcal{A}[G(r)] + g_r)]\right).$$

The analytic discs *per se* are the Poisson integrals of the functions on the right-hand side above. Standard facts about the Poisson integral imply that in order to show that $\mathfrak{g} : (0, R_0) \longrightarrow A^\alpha(\mathbb{D}; \mathbb{C}^2)$ is of class $\mathcal{C}^1$, it suffices to show that G is smooth on $(0, R_0)$.

We remark that the inequalities in Step 3, which culminate in the estimate (3.20), could have been carried out in an exactly analogous manner (i.e. by applying the fundamental theorem of calculus appropriately) with both ψ and r taken to be variable. We refrained from doing this so as to avoid writing out lengthy, but essentially basic, estimates. However, the estimates in Step 3 have been presented sufficiently carefully that we may leave it to the reader to emulate them, and verify that there exist constants $K_1, K_2 > 0$ such that, shrinking $R_0 > 0$ further if necessary:

$$\begin{aligned} \|H(\psi_1; r_1) - H(\psi_2; r_2)\|_{\mathcal{C}^\alpha} \;&\leq\; (1/2)\, \|\psi_1 - \psi_2\|_{\mathcal{C}^\alpha} + \left(K_1 + \frac{K_2}{r_1^{m-1} r_2^{m-1}}\right) |r_1 - r_2| \\ &\forall (\psi_1, r_1), (\psi_2, r_2) \in \{(\psi, r) \in \mathcal{C}^\alpha(\partial\mathbb{D}; \mathbb{R}) \times (0, R_0) : \|\psi\|_{\mathcal{C}^\alpha} \leq r^{1+\delta}\}, \end{aligned} \tag{3.23}$$

where $\delta > 0$ is as chosen in Step 3.

Let us now consider again the map $\Phi : \Omega_\alpha \times (0, 3\rho/4) \longrightarrow \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R})$. We refer back to the beginning of this proof to compute that the total derivative of Φ at the point (ψ, r) has the matrix representation

$$D\Phi(\psi, r) \;=\; [\,\Lambda_r + O(r^m) \quad \partial_r\Phi(\psi_r, r)\,] : \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R}) \oplus \mathbb{R} \longrightarrow \mathcal{C}^\alpha(\partial\mathbb{D};\mathbb{R}),$$

where Λ_r is as defined in Step 2, and $\partial_r\Phi$ denotes the partial Fréchet derivative with respect to r. It is easy to show that the latter exists, and that $D\Phi$ varies continuously with $(\psi, r) \in \Omega_\alpha \times (0, 3\rho/4)$. Shrinking R_0 if necessary, it follows from our Claim in Step 2 that $\partial_\psi\Phi|_{(\psi_r,r)}$ is an isomorphism for each $r \in (0, R_0)$. The reader is now referred to the note at the beginning of Step 2. In view of our last assertion, and the fact that $\Phi(\psi_r, r) = 0$, we can now apply the Implicit Function Theorem. For a given $r^0 \in (0, R_0)$, there exist a $\mathcal{C}^\alpha$-open neighbourhood $\omega(r^0)$ and an interval $I(r^0) \Subset (0, R_0)$ containing r^0 such that:

- For each $r \in I(r^0)$, there exists a *unique* $\psi \in \omega(r^0)$ such that $\Phi(\psi, r) = 0$.
- If we designate this ψ as $\gamma_{r^0}(r)$, then γ_{r^0} is of class $\mathcal{C}^1$ on $I(r^0)$.

We now recall that $H(G(r), r) = G(r)\ \forall r \in (0, R_0)$. Applying this fact to (3.23), we get

$$\frac{|G(r) - G(r^0)|}{2} \;\leq\; \left(K_1 + \frac{K_2}{(r^0)^{m-1}r^{m-1}}\right)|r^0 - r|,$$

whence $\lim_{r\to r^0} G(r) = G(r^0)$. Combining this with the conclusions of the Implicit Function Theorem, there exists an open interval $I'(r^0) \subseteq I(r^0)$ such that

$$G|_{I'(r^0)} \;=\; \gamma_{r^0}|_{I'(r^0)}\,.$$

But as γ_{r^0} is $\mathcal{C}^1$-smooth, and $r^0 \in (0, R_0)$ was picked arbitrarily, we conclude that G is $\mathcal{C}^1$-smooth. Owing to the mapping properties of the Poisson kernel, it follows from Fact B that the Hölder norms, thus the sup-norms of $\mathfrak{g}(r)$, shrink to zero as $r \longrightarrow 0^+$. □

Remark 3.1. It might seem to the reader that since (in the notation of the proof above) $\{(\kappa r\widetilde{G}, (\kappa r)^m) : r \in (0, 1)\}$ is a family of analytic discs with boundaries in $\Gamma(\mathcal{F}_m)\setminus\{(0,0)\}$, we could use Forstnerič's results in [6] to give a "quick" proof of Theorem 1.3. However, the relevant theorems in [6], i.e. Theorems 1 and 3, are *non-quantitative.* We would have to augment them with estimates (in a similar spirit to those in the proof above) to learn whether $\Gamma(\mathcal{F}_m + \mathcal{R}) \setminus \{(0,0)\}$ is "close enough" to $\Gamma(\mathcal{F}_m) \setminus \{(0,0)\}$ for us to deduce the existence of the desired family $\{\mathfrak{g}(r) : r \in (0, R_0)\}$ (especially the existence of $\mathfrak{g}(r)$'s *arbitrarily close* to the CR singularity).

4. A comparison of Theorem 1.3 with previous results

Theorem 1.3 is reminiscent of some of results in [14] about the existence of analytic discs in the polynomially-convex hull around a degenerate CR singularity. We paraphrase Wiegerinck's results to the context that we have been studying.

Result 4.1 (paraphrasing parts of Theorem 3.3 and 3.5, [14])**.** *Let φ be $\mathcal{C}^{m+1}$-smooth function defined in a neighbourhood of $0 \in \mathbb{C}$ that vanishes to order m at 0. Write*

$$\varphi(z) \;=\; \mathcal{F}_m(z) + \mathcal{R}(z) \quad (\textit{with } |z| \textit{ sufficiently small}),$$

where $\mathcal{F}_m$ is polynomial that is homogeneous of degree m, and $\mathcal{R}(z) = O(|z|^{m+1})$. Suppose $(0,0)$ is an isolated CR singularity of $\Gamma(\varphi)$ and that $\mathcal{F}_m$ is real-valued. If $\mathrm{Ind}_{\mathcal{M}}(\Gamma(\varphi), 0) > 0$ and $\mathcal{F}_m$ is a subharmonic, non-harmonic function, then $\Gamma(\varphi)$ is not locally polynomially convex at $(0,0)$.

Results like the above rely strongly on the results of Chirka & Shcherbina [4] (also refer to [11] by Shcherbina), which can be used to analyse the structure of the polynomially-convex hulls of graphs of functions defined on certain classes of sets in $\mathbb{C}^2$ that are homeomorphic to the 2-sphere. The potential-theoretic ideas used in [11] and [4] shape the hypotheses of the results therein. Those hypotheses lead to certain subharmonicity conditions being imposed in the results of [14]. For example, in the setting of Result 4.1, they translate into the requirement that $\mathcal{F}_m$ be a subharmonic, non-harmonic function. This raises the following question: *with the hypotheses imposed on $\mathcal{F}_m$ in Theorem 1.3, is it possible that $\mathcal{F}_m$ is automatically subharmonic?* If this were the case, then Theorem 1.3 would be a special case of the results in [14].

We demonstrate in this section that the answer to the above question is negative. There are pairs $(\mathcal{S}, p)$, where p is an isolated degenerate CR singularity, to which Wiegerinck's hypotheses do not apply but which admit Bishop discs. The point of Theorem 1.3 was to demonstrate some techniques for examining the local polynomially-convex hull near an isolated CR singularity that do not require any subharmonicity-type conditions. We now present a one-parameter family of relevant counterexamples.

Example 4.2. *For each $C \in (1/3, 2/3)$, there exists an $\varepsilon_C > 2/3$ such that the real-valued, homogeneous polynomial*

$$\mathcal{F}_C(z) \ := \ \frac{C}{2}(z^4 + \overline{z}^4) + \varepsilon_C(z^3\overline{z} + z\overline{z}^3) + |z|^4$$

has the following properties:

a) 0 *is an isolated CR singularity of* $\Gamma(\mathcal{F}_C)$ *satisfying* $\text{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_C), 0) > 0$*; and*
b) $\mathcal{F}_m$ *is* not *subharmonic.*

To arrive at a polynomial with the above properties, let us first examine

$$\mathcal{F}(z; \varepsilon, C) \ := \ \frac{C}{2}(z^4 + \overline{z}^4) + \varepsilon(z^3\overline{z} + z\overline{z}^3) + |z|^4.$$

Then

$$\partial^2_{z\overline{z}}\mathcal{F}(z; \varepsilon, C) \ = \ 3\varepsilon(z^2 + \overline{z}^2) + 4|z|^2 \ = \ (6\varepsilon\cos 2\theta + 4)|z|^2,$$

where, as usual, we write $z = |z|e^{i\theta}$. Then, clearly

$$\mathcal{F}(\cdot; \varepsilon, C) \text{ fails to be subharmonic} \iff \varepsilon > 2/3. \tag{4.1}$$

From Lemma 2.4, we realise that

$$\text{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_C), 0) > 0 \iff \mathcal{F}_C(e^{i\theta}) \neq 0 \ \forall\theta \in \mathbb{R}. \tag{4.2}$$

Hence, to begin with, we shall examine whether there are any values of the parameter C such that

$$\mathcal{F}(e^{i\theta}; 2/3, C) \ = \ 2C(\cos 2\theta)^2 + (4/3)\cos 2\theta + (1 - C) \ > \ 0 \ \forall\theta \in \mathbb{R}.$$

We will then perturb the parameter ε away from $\varepsilon = 2/3$ so as to ensure that positivity is preserved, *but subharmonicity fails.* To this end, we set $X := \cos 2\theta$ in the above inequality to get

$$2CX^2 + (4/3)X + (1 - C) \ > 0. \tag{4.3}$$

Note that:

$$\begin{aligned}(4.3) \ &\iff \ \begin{cases} C > 0, \text{ and} \\ (4/3)^2 - 8C(1 - C) < 0. \end{cases} \\ &\iff \ C \in (1/3, 2/3).\end{aligned}$$

This shows that for each $C \in (1/3, 2/3)$, $\mathcal{F}(\bullet; 2/3, C) > 0$ on $\mathbb{C} \setminus \{0\}$.

Finally, note that

- $S^1 \times \{2/3\} \times \{C\}$ is a compact subset of $S^1 \times (\mathbb{R}_+) \times (1/3, 2/3)$; and
- $\mathcal{F}|_{S^1\times\{2/3\}\times\{C\}} > 0$ for each $C \in (1/3, 2/3)$.

Since $\mathcal{F}$ is continuous on $S^1 \times (\mathbb{R}_+) \times (1/3, 2/3)$, there exists a $\delta(C) > 0$ such that

$$(\varepsilon, C) \in \mathbb{B}^2((2/3, C); \delta(C)) \implies \mathcal{F}(e^{i\bullet}; \varepsilon, C) > 0. \tag{4.4}$$

We now pick an $\varepsilon_C \in (2/3, 2/3 + \delta(C))$ and define $\mathcal{F}_C := \mathcal{F}(\bullet; \varepsilon_C, C)$. From (4.2), (4.3) and (4.4), we conclude that $\mathcal{F}_C$ satisfies property (a). We have chosen $\varepsilon_C > 2/3$; hence, by (4.1), $\mathcal{F}_C$ fails to be subharmonic. ∎

5. The proof of Theorem 1.5

A non-trivial result that we will require is the following theorem by Forstnerič, which we shall paraphrase:

Result 5.1 (paraphrasing Theorem 2, [6])**.** *Let M be a maximally totally-real $\mathcal{C}^4$-smooth submanifold of an open subset of $\mathbb{C}^2$, and let $\mathfrak{g} \in A^\alpha(\mathbb{D}; \mathbb{C}^2)$ be an immersed analytic disc with boundary in M such that the tangent bundle TM is trivial over an M-open neighbourhood of $\mathfrak{g}(\partial\mathbb{D})$. If $\operatorname{Ind}_{\mathcal{M},\mathfrak{g}(e^{i\bullet})} \leq 0$, then there is an open neighbourhood $\Omega \subset A^\alpha(\mathbb{D}; \mathbb{C}^2)$ of $\mathfrak{g}$ such that the only analytic discs $F \in \Omega$ with boundary in M are of the form $\mathfrak{g} \circ \varphi$, where $\varphi \in \operatorname{Aut}(\mathbb{D})$.*

Remark 5.2. Theorem 2 in [6] has been stated — in the notation of Result 5.1 — only for $\mathfrak{g} \in A^{1/2}(\mathbb{D}; \mathbb{C}^2)$. However, the observations made in [6, Remark 1] about Theorem 1 apply as well to Theorem 2 in [6]. In other words, we can allow $\mathfrak{g} \in A^\alpha(\mathbb{D}; \mathbb{C}^2)$ in the hypothesis of the latter theorem.

We are now ready to provide

The proof of Theorem 1.5. We first consider Part (1). Let $(\mathcal{S}, p)$ be as described in the hypothesis of the theorem. As before, we may work with the graph $\Gamma(\mathcal{F}_m + \mathcal{R})$, where $\mathcal{F}_m$ and $\mathcal{R}$ have the same meanings as in (3.1). Since $\operatorname{Ind}_{\mathcal{M}}(\mathcal{S}, p)$ is invariant under a holomorphic change of coordinate, arguing exactly as in the proof of Theorem 1.3, $\operatorname{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_m), 0) \leq 0$. By hypothesis, and the formula (2.2) in Lemma 2.4, we conclude that $\mathcal{F}_m$ changes sign. To see this, we rely on the fact that $(0, 0)$ is an isolated CR singularity of $\Gamma(\mathcal{F}_m)$. We have discussed that in this case — see equation (2.3) — if $\mathcal{F}_m(e^{i\bullet})$ has zeros, then it has only simple zeros. This fact — combined with the fact that, by the formula (2.2), $\mathcal{F}_m(e^{i\bullet})^{-1}\{0\} \neq \emptyset$ — implies that $\mathcal{F}_m$ must change sign. Then, each level set $\mathcal{F}_m^{-1}\{c\}$, $c \in \mathbb{R}$, is a finite union of disjoint arcs in $\mathbb{C}$.

Since $\mathcal{R}(z) = O(|z|^{m+1})$, there exists a $\delta > 0$ which is sufficiently small that the level sets of $(\mathcal{F}_m + \mathcal{R})|_{\overline{D(0;\delta)}}$, i.e. the sets

$$\{z \in \overline{D(0;\delta)} : (\mathcal{F}_m + \mathcal{R})(z) = c\} \tag{5.1}$$

do not separate $\mathbb{C}$ for each $c > 0$. We now appeal to the following:

Result 5.3 (Theorem 1.2.16, [12])**.** *If $X \subset \mathbb{C}^n$ is compact and if $\mathcal{P}(X)$ contains a real-valued function f, then X is polynomially convex if and only if each fibre $f^{-1}\{c\}$, $c \in \mathbb{R}$, is polynomially convex.*

We clarify that, for $X \subset \mathbb{C}^n$ compact,

$$\mathcal{P}(X) \ := \ \text{the uniform algebra on } X \text{ generated by the class } \{P|_X : P \in \mathbb{C}[z_1, \dots, z_n]\ \}.$$

Taking $X := \Gamma(\mathfrak{F}_m + \mathcal{R}; \overline{D(0;\delta)})$ and $f(z,w) := w$, and observing that each of the sets in (5.1) is polynomially convex, we conclude from Result 5.3 that $\Gamma(\mathfrak{F}_m + \mathcal{R})$ is locally polynomially convex at $(0,0)$ — or, equivalently, that $\mathcal{S}$ is locally polynomially convex at p.

We now consider Part (2). As before, we shall work in the coordinate system (z,w) with respect to which $(\mathcal{S}, p)$ has the representation (3.1). Suppose, for some $\alpha \in (0,1)$, there exists a continuous one-parameter family $\mathfrak{g} : (0,1) \longrightarrow A^\alpha(\mathbb{D}; \mathbb{C}^2)$ of immersed, non-constant analytic discs with the following three properties:

a) $\mathfrak{g}(t)(\partial\mathbb{D}) \subset (\mathcal{S} \setminus \{p\}) \cap U_p \ \forall t \in (0,1)$.
b) $\mathfrak{g}(t)(e^{i\cdot})$ is a simple closed curve in $\mathcal{S}$ $\forall t \in (0,1)$.
c) $\mathfrak{g}(t)(\zeta) \longrightarrow \{p\}$ for each $\zeta \in \overline{\mathbb{D}}$ as $t \longrightarrow 0^+$.

Let $\delta_0 > 0$ be so small that for every smooth, positively-oriented, simple closed path $\gamma : S^1 \longrightarrow D(0;\delta_0) \setminus \{(0,0)\}$,

$$\mathsf{Wind}\left(\frac{\partial(\mathfrak{F}_m + \mathcal{R})}{\partial \overline{z}} \circ \gamma, 0\right) \ = \ \mathsf{Wind}\left(\frac{\partial \mathfrak{F}_m}{\partial \overline{z}} \circ \gamma, 0\right). \tag{5.2}$$

Such a $\delta_0 > 0$ exists because $\mathcal{R}(z) = O(|z|^{m+1})$. Then, in view of Lemma 2.3 and the properties (a)–(c), (5.2) allows us to infer that there exists a $t_0 \in (0,1)$ such that

$$\begin{aligned} \mathrm{Ind}_{\mathcal{M}, \mathfrak{g}(t)(e^{i\cdot})}(\Gamma(\mathfrak{F}_m + \mathcal{R})) \ &= \ \mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathfrak{F}_m + \mathcal{R}), 0) \\ &= \ \mathrm{Ind}_{\mathcal{M}}(\Gamma(\mathfrak{F}_m), 0) \ \leq \ 0, \ \ \forall t \in (0, t_0). \end{aligned}$$

We pick a $t^* \in (0, t_0)$. Since $\mathcal{S}$ is now assumed to be $\mathcal{C}^4$-smooth, we can apply Result 5.1. By this result, $\exists \varepsilon > 0$ such that for any analytic disc $F \in A^\alpha(\mathbb{D}; \mathbb{C}^2)$ with boundary in $\Gamma(\mathfrak{F}_m + \mathcal{R}) \setminus \{(0,0)\}$ such that $0 < \|F - \mathfrak{g}(t^*)\|_{\mathcal{C}^\alpha} < \varepsilon$, $F = \mathfrak{g}(t^*) \circ \varphi$, where $\varphi \in \mathrm{Aut}(\mathbb{D})$. However, this leads to a contradiction because, owing to the continuity of $\mathfrak{g}$ and to (c) above, there must exist a $t' \in (0, t_0)$, $t' \neq t^*$, such that

$$0 \ < \ \left\|\mathfrak{g}(t^*) - \mathfrak{g}(t')\right\|_{\mathcal{C}^\alpha} \ < \ \varepsilon, \ \text{ and } \mathrm{Image}(\mathfrak{g}(t^*)) \ \neq \ \mathrm{Image}(\mathfrak{g}(t')).$$

Hence, our assumption about the existence of $\mathfrak{g} : (0,1) \longrightarrow A^\alpha(\mathbb{D}; \mathbb{C}^2)$ must be wrong, which establishes Part (2). □

Acknowledgements. A large part of the preliminary work on this article — looking up the extensive literature on the subject, in particular — was done while on a visit to the ICTP, Trieste, during February–March, 2008. The support and hospitality of the ICTP, and the facilities provided by the ICTP library, are warmly acknowledged. I am grateful to Nessim Sibony for suggesting a simple proof of Part (1) of Theorem 1.5 during a visit to the Université Paris-Sud 11 in 2009. This visit was supported by the ARCUS programme (French Foreign Ministry/Région Ile-de-France).

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