# A CRITERION FOR A DEGREE-ONE HOLOMORPHIC MAP TO BE A BIHOLOMORPHISM

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Abstract. Let $X$ and $Y$ be compact connected complex manifolds of the same dimension with $b_2(X) = b_2(Y)$. We prove that any surjective holomorphic map of degree one from $X$ to $Y$ is a biholomorphism. A version of this was established by the first two authors, but under an extra assumption that $\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y)$. We show that this condition is actually automatically satisfied.

## 1. Introduction

Let $X$ and $Y$ be compact connected complex manifolds of dimension $n$. Let

$$f : X \longrightarrow Y$$

be a surjective holomorphic map such that the degree of $f$ is one, meaning that the pullback homomorphism

$$\mathbb{Z} \simeq H^{2n}(Y, \mathbb{Z}) \xrightarrow{f^*} H^{2n}(X, \mathbb{Z}) \simeq \mathbb{Z}$$

is the identity map of $\mathbb{Z}$. It is very natural to ask, "Under what conditions would $f$ be a biholomorphism?" An answer to this was given by [2, Theorem 1.1], namely:

**Result 1** ([2, Theorem 1.1])**.** *Let $X$ and $Y$ be compact connected complex manifolds of dimension $n$, and let $f : X \longrightarrow Y$ be a surjective holomorphic map such that the degree of $f$ is one. Assume that*

*(i) the $\mathcal{C}^\infty$ manifolds underlying $X$ and $Y$ are diffeomorphic, and*

*(ii) $\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y)$.*

*Then, the map $f$ is a biholomorphism.*

In the proof of Result 1, the condition $(i)$ is used *only* in concluding that $\dim H^2(X, \mathbb{R}) = \dim H^2(Y, \mathbb{R})$. In other words, the proof of [2, Theorem 1.1] establishes that if

$$\dim H^2(X, \mathbb{R}) = \dim H^2(Y, \mathbb{R}) \quad \text{and} \quad \dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y),$$

then — with $X$, $Y$, and $f$ as above — $f$ is a biholomorphism.

There is some cause to believe that the condition $(ii)$ in Result 1 might be superfluous (which we shall discuss presently). It is the basis for our main theorem, which gives a simple, purely topological, criterion for a degree-one map to be a biholomorphism:

**Theorem 2.** *Let $X$ and $Y$ be compact connected complex manifolds of dimension $n$, and let $f : X \longrightarrow Y$ be a surjective holomorphic map of degree one. Then, $f$ is a biholomorphism if and only if the second Betti numbers of $X$ and $Y$ coincide.*

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If $X$ and $Y$ were assumed to be Kähler, then Theorem 2 would follow from Result 1. This is because, by the Hodge decomposition, $\dim H^1(M, \mathcal{O}_M) = \frac{1}{2}\dim H^2(M, \mathbb{C})$ for any compact Kähler manifold $M$. We shall show that this observation — i.e., that condition $(ii)$ in Result 1 is automatically satisfied under the hypotheses therein — holds true in the general, *analytic* setting. In more precise terms, we have:

**Proposition 3.** *Let the manifolds $X$ and $Y$ and $f : X \longrightarrow Y$ be as in Result 1. Then, $f$ induces an isomorphism between $H^1(X, \mathcal{O}_X)$ and $H^1(Y, \mathcal{O}_Y)$. In particular, $\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y)$.*

The above proposition might be unsurprising to many. It is well known when $X$ and $Y$ are projective. Since we could not find an explicit statement of Proposition 3 — and since certain supplementary details are required in the analytic case — we provide a proof of it in Section 2. The *non-trivial* step in proving Theorem 2 uses Result 1: given Proposition 3, our theorem follows from Result 1 and the comment above upon its proof.

## 2. Proof of Proposition 3

We begin with a general fact that we shall use several times below. For any proper holomorphic map $F : V \longrightarrow W$ between complex manifolds, the Leray spectral sequence gives the following exact sequence:

$$0 \longrightarrow H^1(W, F_*\mathcal{O}_V) \xrightarrow{\theta_F} H^1(V, \mathcal{O}_V) \longrightarrow H^0(W, R^1F_*\mathcal{O}_V) \longrightarrow \cdots . \tag{2.1}$$

With our assumptions on $X$, $Y$ and $f$, the map $f^{-1}$ (which is defined outside the image in $Y$ of the set of points at which $f$ fails to be a local biholomorphism) is holomorphic on its domain. Thus $f$ is bimeromorphic.

We note that any bimeromorphic holomorphic map of connected complex manifolds has connected fibers, because it is biholomorphic on the complement of a thin analytic subset. In particular, the fibers of $f$ are connected.

**Claim 1.** *Let $F : V \longrightarrow W$ be a bimeromorphic holomorphic map between compact, connected complex manifolds. The natural homomorphism*

$$\mathcal{O}_W \longrightarrow F_*\mathcal{O}_V \tag{2.2}$$

*is an isomorphism.*

By definition, (2.2) is injective. In our case, it is an isomorphism outside a closed complex analytic subset of $W$, say $\mathcal{S}$, of codimension at least 2. So, to show that (2.2) is surjective, it suffices to show that given any $w \in \mathcal{S}$, for each open connected set $U \ni w$ and each holomorphic function $\psi$ on $F^{-1}(U)$ there is a function $H_\psi$ holomorphic on $U$ such that

$$\psi = H_\psi \circ F \quad \text{on } F^{-1}(U).$$

Since $F^{-1}$ is holomorphic on $W \setminus \mathcal{S}$, we set

$$H_\psi|_{U\setminus\mathcal{S}} := \psi \circ (F^{-1}|_{U\setminus\mathcal{S}}).$$

This has a unique holomorphic extension to $U$ by Hartogs' theorem (or more acurately: Riemann's second extension theorem), since $\mathcal{S}$ is of codimension at least 2. As $F$ has compact, connected fibers, this extension has the desired properties. This shows that the homomorphism in (2.2) is surjective. Hence the claim.

By Claim 1, (2.1) yields an injective homomorphism

$$\Theta_f \,:\, H^1(Y,\,\mathcal{O}_Y) \longrightarrow H^1(X,\,\mathcal{O}_X)\,, \tag{2.3}$$

which is the composition of the homomorphism $\theta_f$, as given by (2.1), and the isomorphism induced by (2.2).

There is a commutative diagram of holomorphic maps

$$\begin{array}{ccc} & & Z \\ & \swarrow^{g} & \downarrow h \\ X & \xrightarrow{\;f\;} & Y\,, \end{array} \tag{2.4}$$

where $h$ is a composition of successive blow-ups with smooth centers, such that the subset of $Y$ over which $h$ fails to be a local biholomorphism (i.e., the image in $Y$ of the the exceptional locus in $Z$) coincides with the subset of $Y$ over which $f$ fails to be a local biholomorphism. This fact (also called "Hironaka's Chow Lemma") can be deduced from Hironaka's Flattening Theorem [4, p. 503], [4, p. 504, Corollary 1]. We recollect briefly the argument for this. The set $\mathcal{A}$ of values of $f$ in $Y$ at which $f$ is not flat coincides with the set of points over which $f$ is not locally biholomorphic. Hironaka's Flattening Theorem states that there exists a sequence of blow-ups of $Y$ with smooth centers over $\mathcal{A}$ amounting to a map

$$h \,:\, Z \longrightarrow Y$$

such that — with $\widetilde{Z}$ denoting the proper transform of $Y$ in $X \times_Y Z$ and $\mathsf{pr}_Z$ denoting the projection $X \times_Y Z \longrightarrow Z$ — the map $\widetilde{f} := \mathsf{pr}|_{\widetilde{Z}}$ is flat. In our case this implies that $\widetilde{f} \,:\, \widetilde{Z} \longrightarrow Z$ is a biholomorphism. The map $g = \mathsf{pr}_X \circ (\widetilde{f}\,)^{-1}$ and has the properties stated above.

The maps $h$ and $g$ above are proper modifications. Thus, all the assumptions in Claim 1 hold true for $g \,:\, Z \longrightarrow X$. Hence, we conclude that the homomorphism $\mathcal{O}_X \longrightarrow g_*\mathcal{O}_Z$ is an isomorphism. By (2.1) applied now to $(V, W, F) = (Z, X, g)$, the homomorphism

$$\Theta_g \,:\, H^1(X,\,\mathcal{O}_X) \longrightarrow H^1(Z,\,\mathcal{O}_Z), \tag{2.5}$$

which is analogous to $\Theta_f$ above, is injective.

Similarly, the homomorphism $\mathcal{O}_Y \longrightarrow h_*\mathcal{O}_Z$ is an isomorphism. Since (2.1), an exact sequence, is natural, we would be done — in view of (2.3), (2.5) and the diagram (2.4) — if we show that the homomorphism $\Theta_h \,:\, H^1(Y,\,\mathcal{O}_Y) \longrightarrow H^1(Z,\,\mathcal{O}_Z)$, (given by applying (2.1) to $(V, W, F) = (Z, Y, h)$) is an isomorphism.

To this end, we will use the following:

**Claim 2.** *For a complex manifold $W$ of dimension $n$, if*

$$\sigma \,:\, S \longrightarrow W$$

*is a blow-up with smooth center, then the direct image $R^1\sigma_*\mathcal{O}_S$ vanishes.*

This claim is familiar to many. However, since it is not so easy to point to one *specific* work for a proof in the *analytic* case, we indicate an argument. We first study the blow-up $\widetilde{\sigma} \,:\, \widetilde{S} \longrightarrow \widetilde{W}$ of a point $0 \in \widetilde{W}$ with exceptional divisor $\widetilde{E} = \sigma^{-1}(0)$.

We use the "Theorem on formal functions" [3, Theorem 11.1], and the "Grauert comparison theorem" [1, Theorem III.3.1] for the analytic case. Let $\mathfrak{m}_0 \subset \mathcal{O}_{\widetilde{W}}$ be the maximal ideal sheaf for the point $0 \in \widetilde{W}$. Then the completion $\big((R^1\widetilde{\sigma}_*\mathcal{O}_{\widetilde{S}})_0\big)^{\vee}$ of $(R^1\widetilde{\sigma}_*\mathcal{O}_{\widetilde{S}})_0$ in the $\mathfrak{m}_0$-adic topology is equal to

$$\varprojlim_k H^1\left(\widetilde{\sigma}^{-1}(0),\, \mathcal{O}_{\widetilde{S}}/\widetilde{\sigma}^*(\mathfrak{m}_0^k)\right).$$

We have the exact sequence

$$0 \longrightarrow \mathcal{O}_E(k) \longrightarrow \mathcal{O}_{\widetilde{S}}/\widetilde{\sigma}^*(\mathfrak{m}_0^{k+1}) \longrightarrow \mathcal{O}_{\widetilde{S}}/\widetilde{\sigma}^*(\mathfrak{m}_0^k) \longrightarrow 0$$

of sheaves with support on

$$\widetilde{\sigma}^{-1}(0) \,=\, \widetilde{E} \,\simeq\, \mathbb{P}^{n-1}$$

so that the cohomology groups $H^q(\widetilde{E},\, \mathcal{O}_{\widetilde{E}}(k))$ vanish for all $k \geq 0$, and $q > 0$. In particular the maps

$$H^1(\widetilde{S},\, \mathcal{O}_{\widetilde{S}}/\widetilde{\sigma}^*(\mathfrak{m}_0^{k+1})) \longrightarrow H^1(\widetilde{S},\, \mathcal{O}_{\widetilde{S}}/\widetilde{\sigma}^*(\mathfrak{m}_0^k))$$

are isomorphisms for $k \geq 1$, and furthermore we have

$$H^1(\widetilde{S},\, \mathcal{O}_{\widetilde{S}}/\widetilde{\sigma}^*(\mathfrak{m}_0)) \,\simeq\, H^1(\mathbb{P}^{n-1},\, \mathcal{O}_{\mathbb{P}^{n-1}}) \,=\, 0\,.$$

This shows that $R^1\widetilde{\sigma}_*\mathcal{O}_{\widetilde{S}}$ vanishes. This establishes the claim for blow-up at a point.

Now consider the case where the center of the blow-up $\sigma$ is a smooth submanifold $A$ of positive dimension. Since the claim is local with respect to the base space $W$, we may assume that $W$ is of the form $A \times \widetilde{W}$, where both $A$ and $\widetilde{W}$ are small open subsets of complex number spaces, e.g. polydisks. Denote by $\pi \,:\, W \longrightarrow \widetilde{W}$ the projection. We identify $A$ with $A \times \{0\} = \pi^{-1}(0) \subset W$ as a submanifold.

Note that the blow-up

$$\sigma \,:\, S \longrightarrow W$$

of $W$ along $A$ is the fiber product $\widetilde{S} \times_{\widetilde{W}} W \longrightarrow W$. The exceptional divisor $E$ of $\sigma$ can be identified with $A \times \widetilde{E}$.

In the above argument we replace the maximal ideal sheaf $\mathfrak{m}_0$ by the vanishing ideal $\mathcal{I}_A$ of $A$. Now $\sigma^*(\mathcal{I}_A^k)/\sigma^*(\mathcal{I}_A^{k+1}) \,\simeq\, \mathcal{O}_E(k)$, and by [1, Theorem III.3.4] we have

$$R^1(\sigma|_E)_*\mathcal{O}_E \,\simeq\, \pi^* R^1(\widetilde{\sigma}|_{\widetilde{E}})_*\mathcal{O}_{\widetilde{E}} \,=\, 0$$

so that the earlier argument can be applied. Hence the claim.

Now, let

$$Z \,=\, Z_N \xrightarrow{\tau_N} Z_{N-1} \xrightarrow{\tau_{N-1}} \cdots \xrightarrow{\tau_2} Z_1 \xrightarrow{\tau_1} Z_0 \,=\, Y$$

be the sequence of blow-ups that constitute $h \,:\, Z \longrightarrow Y$. We have $\tau_{j*}\mathcal{O}_{Z_j} \simeq \mathcal{O}_{Z_{j-1}}$ and $R^1\tau_{j\,*}\mathcal{O}_{Z_j} = 0$ for $1 \,\leq\, j \,\leq\, \tau_N$. Combining these with (2.1) yields a canonical injective homomorphism

$$H^1(Z_j,\, \mathcal{O}_{Z_j}) \longrightarrow H^1(Z_{j-1},\, \mathcal{O}_{Z_{j-1}})$$

that is an isomorphism for all $j \,=\, 1, \cdots, N$. Hence, by naturality, the homomorphism $\Theta_h \,:\, H^1(Y,\, \mathcal{O}_Y) \longrightarrow H^1(Z,\, \mathcal{O}_Z)$ is an isomorphism. By our above remarks, this establishes the result.

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