Coalescence in Galton-Watson Trees

K. B. Athreya

Iowa State University, Ames, Iowa. U.S.A.

Jan. 11, 2013
Outline

1. The problem of coalescence in trees
2. Binary tree case
3. Galton-Watson trees
   1) Definition
   2) Basic results
4. Coalescence results for Galton-Watson trees
   a) Supercritical (\(1 < m < \infty\))
   b) Critical (\(m = 1\))
   c) Subcritical (\(0 < m < 1\))
   d) Explosive (\(m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1\))
5. Branching random walks
   i) \(1 < m < \infty\)
   ii) \(m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1\)
6. Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases
The Problem of Coalescence in Trees

Binary Tree Case

Galton-Watson Tree Case

Coalescence results for Galton-Watson trees

Branching random walks

Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases

Outline

1. The Problem of Coalescence in Trees
2. Binary Tree Case
3. Galton-Watson Tree Case
   - Definition
   - Basic results
4. Coalescence results for Galton-Watson trees
   - Supercritical \((1 < m < \infty)\)
   - Critical \((m = 1)\)
   - Subcritical \((0 < m < 1)\)
   - Explosive \((m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1)\)
5. Branching random walks
   - \(1 < m < \infty\)
   - \(m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1\)
6. Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases
1. The Problem of Coalescence in Trees

Let $T$ be a rooted tree. Let $\{v_{n1}, v_{n2}, \cdots, v_{nn}\}$ be the set of vertices at the $n$th level.

Pick two of the $v_{ni}$’s by SRSWOR (simple random sampling without replacement) (assuming $Z_n \geq 2$) and trace their lines of descent back in time till they meet for the first time. Call that generation $X_n$.

$X_n$ is call the \textit{coalescence time}.
1. The Problem of Coalescence in Trees

Problems:

a) Find the distribution of $X_n$.

b) Study its limit as $n \to \infty$.

$X_n$ is also called the generation number of the LCA (Last common ancestor) or MRCA (Most recent common ancestor) etc.

c) Do the same with choosing $k$ vertices out of $Z_n$.

d) Do the same with choosing all $Z_n$ vertices out of $Z_n$.

Clearly, the answers depend on how $\mathcal{T}$ is generated.
Outline

1. The Problem of Coalescence in Trees
2. Binary Tree Case
3. Galton-Watson Tree Case
   - Definition
   - Basic results
4. Coalescence results for Galton-Watson trees
   - Supercritical \((1 < m < \infty)\)
   - Critical \((m = 1)\)
   - Subcritical \((0 < m < 1)\)
   - Explosive \((m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1)\)
5. Branching random walks
   - \(1 < m < \infty\)
   - \(m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1\)
6. Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases
2. Binary Tree Case

Consider a binary tree $\mathcal{T}$ starting with one vertex. The tree looks like

At level $n$, there are $2^n$ vertices, $n = 0, 1, 2, \ldots$.
2. Binary Tree Case

Pick two vertices at level \( n \) by SRSWOR. Trace their lines back till they meet. Call that generation \( X_n \). Then, for \( k = 1, 2, \ldots, n \),

\[
P(X_n < k) = \frac{\binom{2^k}{2} 2^{n-k} 2^{n-k}}{\binom{2^n}{2}} = \frac{2^k (2^k - 1) 2^{n-k} 2^{n-k}}{2^n (2^n - 1)} = \frac{1 - 2^{-k}}{1 - 2^{-n}}
\]

So, \( \lim_{n \to \infty} P(X_n < k) = 1 - 2^{-k}, \ k = 1, 2, \ldots \).

Thus, \( X_n \xrightarrow{d} Geo\left(\frac{1}{2}\right) \).

Similar result is true for any regular \( b \)-nary tree, \( b \geq 2 \).
This suggests that the same must be true for a growing Galton-Watson tree.
Outline

1. The Problem of Coalescence in Trees
2. Binary Tree Case
3. Galton-Watson Tree Case
   - Definition
   - Basic results
4. Coalescence results for Galton-Watson trees
   - Supercritical \((1 < m < \infty)\)
   - Critical \((m = 1)\)
   - Subcritical \((0 < m < 1)\)
   - Explosive \((m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1)\)
5. Branching random walks
   - \(1 < m < \infty\)
   - \(m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1\)
6. Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases
3.1 Definition and the problem

Let \( \{p_j\}_{j \geq 0} \) be a probability distribution on \( \mathbb{N}^+ \equiv \{0, 1, 2, \cdots\} \), \( \{\xi_{n,i} : i \geq 1, n \geq 0\} \) be i.i.d \( \sim \{p_j\}_{j \geq 0} \), and \( Z_0 \) be a positive integer (r.v.),

\[
Z_1 = \sum_{i=1}^{Z_0} \xi_{0,i} \]

and

\[
Z_{n+1} = \begin{cases} 
\sum_{i=1}^{Z_n} \xi_{n,i}, & n \geq 0 \quad \text{if } Z_n > 0 \\
0, & \text{if } Z_n = 0
\end{cases}
\]
3.1 Definition and the problem

Then \( \{Z_n\}_{n \geq 0} \) is called a Galton-Watson branching process with initial population \( Z_0 \) and offspring distribution \( \{p_j\}_{j \geq 0} \), and \( \xi_{n,i} \) is the number of offspring of the \( i \)th individual of the \( n \)th generation.

Now, every individual in the \( n \)th generation, \( n \geq 1 \), can be identified by a finite string

\[
u_n \equiv (i_0, i_1, i_2, \cdots, i_n)\]

meaning that this individual is the \( i_n \)th offspring of the \( u_{n-1} \equiv (i_0, i_1, \cdots, i_{n-1}) \) and \( u_0 = i_0 \) is the number associated with the \( i_0 \)th member of the 0th generation.
3.1 Definition and the problem

Let $A_{n,2} \equiv \{Z_n \geq 2\}$ and $B_n \equiv \{Z_n > 1\}$ be events defined on the space of trees.

Consider the following questions:

3.1 a) Conditioned on $A_{n,2}$, pick two individuals in the $n$th generation by SRSWOR and trace their lines back till they meet. Call that generation $X_{n,2}$.

What is the distribution of $X_{n,2}$?

What happens to it as $n \to \infty$?
3.1 Definition and the problem

3.1 b) Do the same thing with \( k \) choices (\( 2 \leq k < \infty \)) by SRSWOR from the \( n \)th generation. Call the coalescence time \( X_{n,k} \). Ask the same questions.

3.1 c) Do the same thing for the whole population. Call the coalescence time \( Y_n \). Ask the same questions, i.e.,

What is the distribution of \( Y_n \) and what happens to it as \( n \to \infty \)?

K. B. Athreya
3.2 Some basic results for Galton-Watson trees

3.2 i) (Supercritical case) Let $p_0 = 0$, $1 < m = \sum_{j=1}^{\infty} jp_j < \infty$.

Then

a) $P(Z_n \to \infty | Z_0 > 0) = 1$.

b) (Harris, 1960)

$$\left\{ W_n \equiv \frac{Z_n}{m^n} : n \geq 0 \right\}$$

is a nonnegative martingale and hence

$$\lim_{n \to \infty} W_n \equiv W$$

exists w.p.1.
3.2 Some basic results for Galton-Watson trees

3.2 i) (Supercritical case) Let \( p_0 = 0, \ 1 < m = \sum_{j=1}^{\infty} j p_j < \infty. \)

Then

c) (Kesten and Stigum, 1966)

\[
\sum_{j=1}^{\infty} (j \log j) p_j < \infty \quad \text{iff} \quad E(W|Z_0 = 1) = 1
\]

and then \( W \) has an absolutely continuous distribution on \((0, \infty)\) with a positive density.

d) (Seneta and Heyde, 1970)

\[
\exists C_n \ni \frac{C_{n+1}}{C_n} \to m \quad \text{and} \quad \frac{Z_n}{C_n} \to W \text{ w.p.1}
\]

and \( P(0 < W < \infty) = 1. \)
3.2 Some basic results for Galton-Watson trees

3.2 i) (Supercritical case) Let $p_0 = 0$, $1 < m = \sum_{j=1}^{\infty} j p_j < \infty$. Then

\[ E(W : W \leq x) \equiv L(x) \]

is slowly varying at $\infty$. 

e) (Athreya and Schuh, 2003)
3.2 Some basic results for Galton-Watson trees

3.2 ii) (Critical case) Let \( m \equiv \sum_{j=1}^{\infty} j p_j = 1 \), \( p_j \neq 1 \) for any \( j \geq 1 \)

and \( \sigma^2 \equiv \sum_{j=1}^{\infty} j^2 p_j - 1 < \infty \). Then

a) \( P(Z_n \to 0 | Z_0 > 0) = 1 \).

b) (Kolmogrov, 1938)

\[ nP(Z_n > 0) \to \frac{\sigma^2}{2} \quad \text{as } n \to \infty. \]

c) (Yaglom, 1947)

\[ P\left(\frac{Z_n}{n} > x \left| Z_n > 0\right.\right) \to e^{-\frac{2}{\sigma^2}x} \quad , \quad 0 < x < \infty. \]
3.2 Some basic results for Galton-Watson trees

3.2 ii)

\[ d) \text{ (Athreya, 2010) For } 1 \leq k \leq n, \text{ let} \]

\[ V_{n,k} \equiv \left\{ \frac{Z_{n-k,i}^{(k)}}{n-k} I(Z_{n-k,i}^{(k)}>0) : 1 \leq i \leq Z_k \right\} \]

on the event \( \{ Z_k > 0 \} \), where \( \{ Z_{j,i}^{(k)} : j \geq 0 \} \) is the G-W process initiated by the \( i \)th individual in the \( k \)th generation. Let \( k \to \infty, n \to \infty \) such that \( \frac{k}{n} \to u, \ 0 < u < 1 \). Then the sequence of point processes \( \{ V_{n,k} \}_{n \geq 1} \) conditioned on \( \{ Z_n \geq 1 \} \) converges weakly to the point process

\[ V \equiv \{ \eta_j : j = 1, 2, \cdots, N_u \} \]

where \( \{ \eta_j \}_{j \geq 1} \) are i.i.d. \( \exp(1) \), \( N_u \) is \( \text{Geom}(u) \), i.e.,

\[ P(N_u = k) = (1 - u)u^{k-1}, \ k \geq 1 \text{ and } \{ \eta_j \}_{j \geq 1} \text{ and } N_u \text{ are independent.} \]
3.2 Some basic results for Galton-Watson trees

3.2 iii) (Subcritical case) (Yaglom, 1947) Let \( 0 < m \equiv \sum_{j=1}^{\infty} j p_j < 1 \).

Then

\[ a) \text{ For } j \geq 1, \lim_{n \to \infty} P(Z_n = j | Z_n > 0) \equiv b_j \text{ exists, } \sum_{j=0}^{\infty} b_j = 1 \]

and \( B(s) \equiv \sum_{j=0}^{\infty} b_j s^j, 0 \leq s \leq 1 \) is the unique solution of the functional equation

\[ B(f(s)) = mB(s) + (1 - s) , 0 \leq s \leq 1 \]

where \( f(s) \equiv \sum_{j=0}^{\infty} p_j s^j \), in the class of all probability generating functions vanishing at 0.
3.2 Some basic results for Galton-Watson trees

3.2 iii) (Subcritical case) (Yaglom, 1947) Let \( 0 < m \equiv \sum_{j=1}^{\infty} j p_j < 1 \).

Then

\[ \sum_{j=1}^{\infty} j b_j < \infty \quad \text{iff} \quad \sum_{j=1}^{\infty} (j \log j) p_j < \infty. \]

\[ \lim_{n \to \infty} \frac{P(Z_n > 0 | Z_0 = 1)}{m^n} = \frac{1}{\sum_{j=1}^{\infty} j b_j}. \]
3.2 iii) (Subcritical case) Let $0 < m \equiv \sum_{j=1}^{\infty} jp_j < 1$. Let $Z_0$ be a random variable. Then

d) If $EZ_0 < \infty$, then

$$\lim_{n \to \infty} P(Z_n = j | Z_n > 0) = b_j, \forall j \geq 1$$

and if, in addition, $\sum_{j=1}^{\infty}(j \log j)p_j < \infty$ then

$$\sum_{j=1}^{\infty} jb_j < \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{P(Z_n > 0)}{m^n} = \frac{EZ_0}{\sum_{j=1}^{\infty} jb_j}.$$
The Problem of Coalescence in Trees

Binary Tree Case

Galton-Watson Tree Case

Coalescence results for Galton-Watson trees

Branching random walks

Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases

Outline

1 The Problem of Coalescence in Trees

2 Binary Tree Case

3 Galton-Watson Tree Case

- Definition
- Basic results

4 Coalescence results for Galton-Watson trees

- Supercritical \((1 < m < \infty)\)
- Critical \((m = 1)\)
- Subcritical \((0 < m < 1)\)
- Explosive \((m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1)\)

5 Branching random walks

- \(1 < m < \infty\)
- \(m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1\)

6 Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases

K. B. Athreya
4.1 Supercritical case

Theorem 4.1:

(Supercritical case) Let $p_0 = 0$, $1 < m \equiv \sum_{j=1}^{\infty} jp_j < \infty$. Then, for almost all trees $T$,

i) for $\forall 1 \leq k < \infty$,

$$\lim_{n \to \infty} P(X_{n,2} < k|T) \equiv \pi_{k,2}(T) \text{ exists}$$

and $\pi_{k,2}(T) \uparrow 1$ as $k \uparrow \infty$. 

K. B. Athreya
4.1 Supercritical case

Theorem 4.1:

**Theorem**

ii) for \( \forall j \geq 2, \forall 1 \leq k < \infty \),

\[
\lim_{n \to \infty} P(X_{n,j} < k | T) \equiv \pi_{k,j}(T) \quad \text{exists}
\]

and \( \pi_{k,j}(T) \uparrow 1 \) as \( k \uparrow \infty \).

iii) Let \( p_1 > 0 \). Then, for almost all trees \( T \),

\[
Y_n \to N(T)
\]

where \( N(T) = \max\{j \geq 1 : Z_j = 1\} \). Also,

\[
\lim_{n \to \infty} P(Y_n = k) = (1 - p_1)p_1^k, \quad k \geq 0.
\]
4.2 Critical case

Theorem 4.2:

(Critical case) Let $m = 1$, $p_1 < 1$ and $\sigma^2 = \sum_{j=1}^{\infty} j^2 p_j - 1 < \infty$, then, for $0 < u < 1$,

\[
\lim_{n \to \infty} P\left( \frac{X_{n,2}}{n} \bigg| Z_n \geq 2 \right) \equiv H_2(u) \text{ exists and for } 0 < u < 1,
\]

\[
H_2(u) \equiv 1 - E\varphi(N_u)
\]

where $N_u$ is a geometric random variable with distribution

\[
P(N_u = k) = (1 - u)u^{k-1}, \quad k \geq 1
\]
4.2 Critical Case

Theorem 4.2:

Theorem

i) (continued) and for \( j \geq 1 \),

\[
\varphi(j) \equiv E\left( \frac{\sum_{i=1}^{j} \eta_i^2}{(\sum_{i=1}^{j} \eta_i)^2} \right)
\]

where \( \{\eta_i\}_{i \geq 1} \) are i.i.d. exponential r.v. with \( E\eta_1 = 1 \).

Further, \( H_2(\cdot) \) is absolutely continuous on \([0, 1]\), \( H_2(0+) = 0 \), and \( H_2(1-) = 1 \).
4.2 Critical Case

Theorem 4.2:

Theorem

ii) for $0 < u < 1$, $1 < k < \infty$,

$$\lim_{n \to \infty} P \left( \frac{X_{n,k}}{n} < u \mid Z_n \geq k \right) \equiv H_k(u) \text{ exists}$$

and $H_k(\cdot)$ is an a.c. distribution function with $H_k(0^+) = 0$ and $H_k(1^-) = 1$.

iii) for $0 < u < 1$, $\lim_{n \to \infty} P \left( \frac{Y_n}{n} < u \mid Z_n \geq 1 \right) = u$.

Remark: iii) above is also proved in Zubkov (1974) (TPA).
4.3 Subcritical case

Theorem 4.3:

(Subcritical case) Let $0 < m \equiv \sum_{j=1}^{\infty} j p_j < 1$. Then

i) For $k \geq 1$, $\lim_{n \to \infty} P(n - X_n > k | Z_n \geq 2) = \frac{E \phi_k(Y)}{E \psi_k(Y)} \equiv \pi_k$, say, where

$$\phi_k(j) = E \left( \frac{\sum_{i_1 \neq i_2=1}^{j} Z_{k,i_1} Z_{k,i_2}}{(\sum_{i=1}^{j} Z_{k,i}) (\sum_{i=1}^{j} Z_{k,i} - 1)} I(\sum_{i=1}^{j} Z_{k,i} \geq 1) \right)$$

K. B. Athreya
4.3 Subcritical case

Theorem 4.3:

i) (continued) and

\[ \psi_k(j) = P \left( \sum_{i=1}^{j} Z_{k,i} \geq 2 \right) \]

where \( \{Z_{r,i} : r \geq 0\}, i = 1, 2, \cdots \) are i.i.d. copies of a Galton-Watson branching process \( \{Z_r : r \geq 0\} \) with \( Z_0 = 1 \) and the given offspring distribution \( \{p_j\}_{j \geq 0} \) and \( Y \) is a random variable with distribution \( \{b_j\}_{j \geq 1} \) where

\[ b_j \equiv \lim_{n \to \infty} P(Z_n = j \mid Z_n > 0, Z_0 = 1) \] which exists.
4.3 Subcritical case

Theorem 4.3:

Theorem

i) (continued) Further, if \( \sum_{j=1}^{\infty} j \log j p_j < \infty \), then \( \lim_{k \uparrow \infty} \pi_k = 0 \) and hence \( n - X_n \) conditioned on \( Z_n \geq 2 \) converges to a proper distribution on \( \{1, 2, \cdots\} \).

ii) For \( k \geq 1 \), \( \lim_{n \to \infty} P(n - Y_n > k \mid Z_n \geq 1) \equiv \tilde{\pi}_k \) exists and equals

\[
E\left(\frac{1 - q_k^Y}{m^k}\right) - E\left(\frac{Y q^{k-1}(1 - q_k)}{m^k}\right)
\]
4.3 Subcritical case

Theorem 4.3:

Theorem

\[ P(Y = j) = b_j = \lim_{n \to \infty} P(Z_n = j|Z_n > 0, Z_0 = 1) \]

and \[ q_k = P(Z_k = 0|Z_0 = 1). \]

Further, if \[ \sum_{j=1}^{\infty} j \log j p_j < \infty, \] then \[ \lim_{k \to \infty} \tilde{\pi}_k = 0. \] That is, \[ n - Y_n \text{ conditioned on } \{Z_n > 0\} \text{ converges in distribution as } n \to \infty \text{ to a proper distribution on } \{1, 2, \cdots\}. \]

4.4 Explosive case

Theorem 4.4:

(Explosive case) Let $p_0 = 0$, $m = \sum_{j=1}^{\infty} j p_j = \infty$, and for some

$0 < \alpha < 1$, and a function $L : (1, \infty) \rightarrow (0, \infty)$ slowly varying at

$\infty$, i.e., $\forall 0 < c < \infty$,

$$\frac{L(cx)}{L(x)} \rightarrow 1 \quad \text{as} \quad x \rightarrow \infty.$$  

Let

$$\sum_{j>x} \frac{p_j}{x^\alpha L(x)} \rightarrow 1 \quad \text{as} \quad x \rightarrow \infty.$$
4.4 Explosive case

Theorem 4.4:

Theorem (continued) Then

i) (Davies, 1979) $\alpha^n \log Z_n \rightarrow \eta$ w.p.1 and $P(0 < \eta < \infty) = 1$
and $\eta$ has a continuous distribution.

ii) (Grey, 1980) Let $\{Z_n^{(1)}\}_{n \geq 1}$ and $\{Z_n^{(2)}\}_{n \geq 1}$ be two i.i.d. copies of a GWBP with $\{p_j\}_{j \geq 1}$ satisfying the above hypotheses. Then, w.p.1

\[
\frac{Z_n^{(1)}}{Z_n^{(2)}} \rightarrow \begin{cases} 
0 & \text{with prob. } \frac{1}{2} \\
\infty & \text{with prob. } \frac{1}{2}
\end{cases}
\]
4.4 Explosive case

Theorem 4.4:

(continued)

(iii) For almost all trees \( T \) and \( k = 1, 2, \ldots \), as \( n \to \infty \),

\[ P(X_{n,2} < k | T) \to 0 \]

and

\[ P(n - X_{n,2} < k) \to \pi_2(k) \text{ exists} \]

and \( \pi_2(k) \uparrow 1 \) as \( k \uparrow \infty \).
4.4 Explosive case

Theorem 4.4:

Theorem

(continued)

iv) For any $1 < j < \infty$ and $k = 1, 2, \ldots$

$$P(X_{n,j} < k | T) \to 0 \quad \text{as } n \to \infty$$

and $P(n - X_{n,j} < k) \to \pi_j(k)$ exists and $\pi_j(k) \uparrow 1$ as $k \uparrow \infty$.

v) $Y_n \xrightarrow{d} N(T) \equiv \max\{j : Z_j = 1\} < \infty$ and

$$P(Y_n = k) \to (1 - p_1)p_1^{k-1}, \quad k \geq 1.$$
Proposition 4.1

The proof of Theorem 4.4 ($m = \infty$ explosive case) needs the following results.

Proposition

Let $\{Z_n\}_{n \geq 0}$ be a GWBP with offspring distribution $\{p_j\}_{j \geq 0} \in D(\alpha)$, (domain of attraction of a stable law of order $\alpha$), $0 < \alpha < 1$, and $Z_0 = 1$. Then,

$$Z_k \in D(\alpha^k), \ \forall 1 \leq k < \infty.$$
Proposition 4.2

Let \( \{X_i\}_{i \geq 1} \) be i.i.d. random variables s.t. \( P(0 < X_1 < \infty) = 1 \) and \( X_1 \in D(\alpha), 0 < \alpha < 1 \). Then

a)
\[
\frac{\sum_{i=1}^{n} X_i^2}{\left( \sum_{i=1}^{n} X_i \right)^2} \xrightarrow{d} Y_\alpha
\]

where \( Y_\alpha \) is a continuous r.v. with \( P(0 < Y_\alpha < 1) = 1 \).
Proposition 4.2

(continued)

b) \( EY_\alpha \uparrow 1 \) as \( \alpha \downarrow 0 \).

c) For any \( j = 2, 3, \ldots \),

\[
\frac{\sum_{i=1}^{n} X_i^j}{\left( \sum_{i=1}^{n} X_i \right)^j} \xrightarrow{d} Y_{\alpha,j}
\]

and \( EY_{\alpha,j} \uparrow 1 \) as \( \alpha \downarrow 0 \).
Basic Calculation

\[ P(X_n \geq k | T) = \frac{\sum_{i=1}^{Z_k} \binom{Z_{n-k,i}^{(k)}}{2}}{(Z_n/2)} \]

\[ = \frac{\sum_{i=1}^{Z_k} Z_{n-k,i}^{(k)} \left( Z_{n-k,i}^{(k)} - 1 \right)}{\left( \sum_{i=1}^{Z_k} Z_{n-k,i}^{(k)} \right) \left( \sum_{i=1}^{Z_k} Z_{n-k,i}^{(k)} - 1 \right)} \]  

\[ (*) \]

K. B. Athreya
Basic Calculation

\[ a) \; 1 < m < \infty \]

Fix \( k \), by Seneta-Heyde, \( \exists C_n \ni Z_k(n-k,1)m^{n-k} \rightarrow W_{k,i} \text{ w.p.} 1 \)

and \( P(0 < W_{k,i} < \infty ) = 1 \). So,

\[ (*) \rightarrow \left( \frac{1}{Z_k} \sum_{i=1}^{Z_k} W_{k,i} \right)^2 \]

and this converges to 0 as \( k \rightarrow \infty \) by O’Brien’s theorem (1980):
Basic Calculation

a) (continued)
Let \( \{X_i\}_{i \geq 1} \) be i.i.d. positive random variables s.t. 
\( E(X_1 : X_1 \leq x) \) is slowly varying at \( \infty \). Then

\[
\max_{1 \leq i \leq n} \frac{X_i}{n} \xrightarrow{p} 0.
\]

\[
\sum_{i=1}^{n} X_i \xrightarrow{p} 0.
\]

K. B. Athreya
The Problem of Coalescence in Trees
Binary Tree Case
Galton-Watson Tree Case
Coalescence results for Galton-Watson trees
Branching random walks
Scaling limits of Bellman-Harris Processes with age dep

Basic Calculation

b) \( m = \infty, \{p_j\} \in D(\alpha), \ 0 < \alpha < 1. \)

\[
P(n - X_n \leq k) = P(X_n \geq n - k) = E\left( \frac{\sum_{i=1}^{Z_{n-k}} Z_{k,i}^{(n-k)} (Z_{k,i}^{(n-k)} - 1)}{Z_n (Z_n - 1)} \right) \rightarrow \pi(k) \equiv E(Y_{\alpha,k})
\]

by Lepage, Woodroff and Zinn, and \( \pi(k) \uparrow 1 \) as \( k \uparrow \infty \) and \( E(Y_\alpha) \uparrow 1 \) as \( \alpha \downarrow 0. \)

c) Similar argument for \( m = 1 \) and \( 0 < m < 1. \) (need point process result for \( m = 1 \) and the Yaglom theorem for \( 0 < m < 1). \)
Summary

$1 < m < \infty$: $X_{n,2} \xrightarrow{d} \text{a proper distribution on } \{0, 1, 2, \cdots \}$

$m = \infty$, $\{p_j\}_{j \geq 0} \in D(\alpha)$, $0 < \alpha < 1$: $n - X_{n,2} \xrightarrow{d} \text{a proper distribution on } \{0, 1, 2, \cdots \}$

$m = 1$, $\sigma^2 < \infty$: $\frac{X_{n,2}}{n} \bigg| Z_n \geq 2 \xrightarrow{d} \text{a.c. distribution on } [0, 1]

\frac{Y_n}{n} \bigg| Z_n \geq 1 \xrightarrow{d} \text{uniform distribution on } [0, 1]$

$0 < m < 1$: $(n - X_{n,2}) \bigg| Z_n \geq 2 \xrightarrow{d} \text{a proper distribution on } \{1, 2, \cdots \}$
Summary

i.e.

1 < m < ∞: coalescence is near the beginning of the tree.

m = ∞, \( \{p_j\}_{j \geq 0} \in D(\alpha) \), 0 < α < 1: coalescence is near the present.

m = 1, \( \sigma^2 < \infty \): \( X_{n,2} \) is of order \( n \).

0 < m < 1: \( X_{n,2} \) is near the present.
The Problem of Coalescence in Trees
  Binary Tree Case
  Galton-Watson Tree Case
Coalescence results for Galton-Watson trees
  Branching random walks

Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases

Outline

1 The Problem of Coalescence in Trees
2 Binary Tree Case
3 Galton-Watson Tree Case
  • Definition
  • Basic results
4 Coalescence results for Galton-Watson trees
  • Supercritical ($1 < m < \infty$)
  • Critical ($m = 1$)
  • Subcritical ($0 < m < 1$)
  • Explosive ($m = \infty$, $\{p_j\} \in D(\alpha)$, $0 < \alpha < 1$)
5 Branching random walks
  • $1 < m < \infty$
  • $m = \infty$, $\{p_j\} \in D(\alpha)$, $0 < \alpha < 1$
6 Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases

K. B. Athreya
5. Branching Random Walks

Let $T$ be a G-W tree with $Z_0 = 1$ and offspring distribution $\{p_j\}_{j \geq 0}$.

Impose on this tree $T$ the following movement structure:

If an individual is at $x$ in $\mathbb{R}$ and has $k$ children then these $k$ children move to $x + X_{k,j}$, $j = 1, 2, \cdots, k$, where $X_k \equiv (X_{k,1}, X_{k,2}, \cdots, X_{k,k})$ has a joint distribution $\pi_k(\cdot)$ on $\mathbb{R}^k$.

Also, the random vector $X_k$ is stochastically independent of the history up to that generation as well as the movement of the other individuals of that generation.
5. Branching Random Walks

Let $Z_n$ be the number of individuals in the $n$th generation and $\zeta_n \equiv \{x_{n,i} : 1 \leq i \leq Z_n\}$ be the positions of the $Z_n$ individuals of the $n$th generation.

A problem of interest is what happens to the point process $\zeta_n$ as $n \to \infty$. 
Theorem 5.1

Let \( p_0 = 0 \), \( 1 < m \equiv \sum_{j=1}^{\infty} jp_j < \infty \) and \( \pi_k \) be such that \{\( X_{k,i} : i = 1, 2, \cdots, k \}\}_{k \geq 1} \) are identically distributed.

a) Let \( E X_{k,1} = 0 \) and \( E X_{k,1}^2 = \sigma^2 < \infty \). Then, \( \forall y \in \mathbb{R}, \)

\[
\frac{Z_n(\sqrt{n}\sigma y)}{Z_n} \rightarrow \Phi(y) \quad \text{(the standard } N(0, 1) \text{ cdf)}
\]

in mean square.
Theorem 5.1

(continued)

b) If \( X_{k,1} \in D(\alpha) \), \( 0 < \alpha \leq 2 \), then \( \exists a_n, b_n \in \mathbb{R} \) such that

\[
\frac{Z_{a_n + b_n y}}{Z_n} \rightarrow G_\alpha(y) \quad \text{in mean square,}
\]

where \( G_\alpha(\cdot) \) is a standard stable law cdf (of order \( \alpha \)).

c) In a), if \( Y_n \) is the position of a randomly chosen individual from the \( n \)th generation, then, \( \forall y \in \mathbb{R} \),

\[
P(Y_n \leq \sqrt{n} \sigma y) \rightarrow \Phi(y)
\]

and similarly for b).
Theorem 5.1

The proof depends on the fact when $p_0 = 0$ and

$$1 < m = \sum_{j=1}^{\infty} j p_j < \infty,$$

the coalescence time $X_{n,2}$ is way back in time and so the positions of two randomly chosen individuals in the $n$th generation are essentially independent and has the marginal distribution of a random walk at step $n$. 

K. B. Athreya
Theorem 5.2

(Athreya-Hong, 2011)

Let \( m = \infty, \{p_j\}_{j \geq 0} \in D(\alpha), 0 < \alpha < 1 \). Let \( \{X_{k,i} : 1 \leq i \leq k\}_{k \geq 1} \) be identically distributed. Let \( EX_{k,1} = 0 \) and \( EX_{k,1}^2 = \sigma^2 < \infty \). Then

\[
\frac{Z_n(\sqrt{n}\sigma y)}{Z_n} \xrightarrow{d} \delta_y
\]

where \( \delta_y \) is Bernoulli(\( \Phi(y) \)), i.e.

\[
\delta_y = \begin{cases} 
1, & \text{with prob. } \Phi(y) \\
0, & \text{with prob. } 1 - \Phi(y)
\end{cases}
\]
Theorem 5.2

The proof is based on the fact that

$$E \left( \frac{Z_n(\sqrt{n}\sigma y)}{Z_n} \right)^k \rightarrow \Phi(y) \quad \text{for } k = 1, 2.$$  

This, in turn, is due to the fact that $X_{n,2}$, the coalescence time for any two individuals chosen at random from the $n$th generation is such that $n - X_{n,k}$ converges to a proper distribution (Theorem 4.4) and hence their positions differ only by an amount that converges in distribution.

This can be strengthened to joint convergence of

$$\frac{Z_n(\sqrt{n}\sigma y)}{Z_n}, \quad i = 1, 2, \ldots, k$$
The Problem of Coalescence in Trees
Binary Tree Case
Galton-Watson Tree Case
Coalescence results for Galton-Watson trees
Branching random walks
Scaling limits of Bellman-Harris Processes with age dep

1 \leq m < \infty
m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1

Theorem 5.3

(Athreya-Hong, 2011)

Under the hypothesis of Theorem 5.2,

a) for any $-\infty < y_1 < y_2 < \infty$,

\[
\left(\frac{Z_n(\sqrt{n\sigma y_1})}{Z_n}, \frac{Z_n(\sqrt{n\sigma y_2})}{Z_n}\right) \overset{d}{\longrightarrow} \left(\delta_1(\Phi(y_1)), \delta_2(\Phi(y_2))\right)
\]

which takes values (0, 0), (0, 1) and (1, 1) with probabilities $1 - \Phi(y_2)$, $\Phi(y_2) - \Phi(y_1)$ and $\Phi(y_1)$, respectively.
Theorem 5.3

(continued)

b) for any $-\infty < y_1 < y_2 < \cdots < y_k < \infty$,

\[
\left( \frac{Z_n(\sqrt{n}\sigma y_i)}{Z_n} : 1 \leq i \leq k \right) \xrightarrow{d} (\delta_1, \cdots, \delta_k)
\]

where each $\delta_i$ is 0 or 1 and further $\delta_i = 1 \Rightarrow \delta_j = 1$ for $j \geq i$ and

\[
P(\delta_1 = 0, \delta_2 = 0, \cdots, \delta_{j-1} = 0, \delta_j = 1, \cdots, \delta_k = 1) = P(\delta_{j-1} = 0, \delta_j = 1) = \Phi(y_j) - \Phi(y_{j-1}).
\]
Theorem 5.3

This suggests that

$$\left\{ Z_n(y) = \frac{Z_n(\sqrt{n} \sigma y)}{Z_n}, -\infty < y < \infty \right\}$$

converges in the Skorohod Space $D(-\infty, \infty)$ weakly to

$$\left\{ X(y) \equiv I_{N \leq y}, -\infty < y < \infty \right\}$$

where $N$ is a $N(0, 1)$ r.v.

This needs to be proved. Only tightness needs to be established.
Theorem 5.4

If $Y_n$ is the position of a randomly chosen individual in the $n$th generation, then in all cases (as long as $p_0 = 0$), given the tree (random walk) $T$, $\forall y \in \mathbb{R}$,

$$P(Y_n \leq \sqrt{n}\sigma y | T) \xrightarrow{d} \delta_y \sim Ber(\Phi(y))$$

This is so since

$$P(Y_n \leq \sqrt{n}\sigma y | T) = \frac{Z_n(\sqrt{n}\sigma y)}{Z_n}$$

and this in turn implies, $\forall y \in \mathbb{R}$,

$$P(Y_n \leq \sqrt{n}\sigma y) \rightarrow \Phi(y).$$
Remark 1

Theorem 5.1 holds under the following weaker assumption about $\pi_k$, the distribution of $(X_{k,1}, X_{x,2}, \cdots, X_{k,k})$, that does not require $\{X_{k,1}\}_{k \geq 1}$ to be identically distributed. It suffices to assume:

i) $\forall k \geq 1$, $(X_{k,1}, X_{x,2}, \cdots, X_{k,k})$ has a distribution that is invariant under permutation.

ii) If $\{p_k\}_{k \geq 1}$ is the offspring distribution with

$$\sum_{k=1}^{\infty} p_k E X_{k,1}^2 < \infty, \quad 1 < m = \sum_{k=1}^{\infty} k p_k < \infty, \quad p_0 = 0.$$
Theorem 5.1'

Now let \( \mu = \sum_{k=1}^{\infty} p_k EX_{k,1} < \infty \), \( \sigma^2 = \sum_{k=1}^{\infty} p_k EX_{k,1}^2 - \mu^2 \).

Theorem

Let \( \zeta_n \equiv \{x_{n,1}, x_{n,2}, \cdots, x_n, Z_n\} \) be as in Theorem 5.1. Under the above assumptions, the following holds: for \( \forall y \in \mathbb{R} \),

\[
\frac{Z_n(n\mu + y\sigma \sqrt{n})}{Z_n} \equiv \frac{1}{Z_n} \sum_{i=1}^{Z_n} I(x_{n,i} \leq n\mu + y\sigma \sqrt{n}) \to \Phi(y) \quad \text{in mean square.}
\]
Application to energy cascades

Consider a particle that undergoes fission.

Assume each particle splits into a random number of new particles with distribution \( \{p_k\}_{k \geq 1} \).

Assume that the energy \( x \) of the parent is split to \( \{xY_{k,1}, xY_{k,2}, \ldots, xY_{k,k}\} \) for each of the offspring particle if the parent splits into \( k \) offspring particles.
Application to energy cascades

Then the energy $e_{n,I_n}$ of a particle $I_n$ in the $n$th generation can be represented as

$$x_0 Y_{u_1} Y_{u_2} \cdots Y_{u_n}$$

where $u_n, u_{n-1}, \cdots, u_1$ are the addresses of the individual $I_n$ and its ancestors and $x_0$ is the energy of the ancestor 1.

Assume $Y_{u_i}$'s are independent. Clearly, the distribution of $Y_{u_i}$ depends on the number of offspring of individual $u_{i-1}$ and

$$\left\{ \log e_{n,I_n}, I_n \in n\text{th generation} \right\}$$

is a branching random walk.
So, from Theorem 5.1', one gets the following.

Theorem

Let \( \{X_{k,i} \equiv \log Y_{k,i} : 1 \leq i \leq k\}_{k \geq 1} \) and \( \{p_k\}_{k \geq 1} \) satisfy the conditions of Theorem 5.1'. Then, \( \forall y \in \mathbb{R} \), as \( n \rightarrow \infty \),

\[
\frac{Z_n(n\mu + y\sigma \sqrt{n})}{Z_n} \equiv \frac{1}{Z_n} \sum_{i=1}^{Z_n} I(\log e_{n,i} \leq n\mu + y\sigma \sqrt{n})
\]

\( \rightarrow \Phi(y) \) in mean square.
Open Cases:

$m = 1$ and $0 < m < 1$. 

$1 < m < \infty$

$m = \infty$, $\{p_j\} \in D(\alpha), \quad 0 < \alpha < 1$
Outline

1. The Problem of Coalescence in Trees
2. Binary Tree Case
3. Galton-Watson Tree Case
   - Definition
   - Basic results
4. Coalescence results for Galton-Watson trees
   - Supercritical \((1 < m < \infty)\)
   - Critical \((m = 1)\)
   - Subcritical \((0 < m < 1)\)
   - Explosive \((m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1)\)
5. Branching random walks
   - \(1 < m < \infty\)
   - \(m = \infty, \{p_j\} \in D(\alpha), 0 < \alpha < 1\)
6. Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases

K. B. Athreya
Scaling Limits of B-H processes with age dependent Markov motion

Suppose we are given:

i) an offspring distribution \( \{p_j\}_{j \geq 1} \) on \( \mathbb{N}^+ \equiv \{0, 1, 2, \cdots\} \)

ii) a lifetime distribution \( G(\cdot) \) on \( (0, \infty) \) and non-latice

iii) a real-valued Markov process \( \eta(\cdot) \) on \( [0, \infty) \) with \( \eta(0) = 0 \)

First, generate a BH tree \( T \) with offspring distribution \( \{p_j\}_{j \geq 0} \) and lifetime distribution \( G(\cdot) \) and an initial population at \( t = 0 \) of size \( Z_0 \).

Now, suppose that the initial population is located at \( x_{0,i}, i = 1, 2, \cdots, Z_0 \) and with ages \( a_{0,i}, i = 1, 2, \cdots, Z_0 \).
Scaling Limits of B-H processes with age dependent Markov motion

Assume each individual moves during its lifetime of length $L$ according to Markov process $\{x + \eta(t) : 0 \leq t \leq L\}$.

That is, if an individual is born at time $\tau$ and at location $x$ and has lifetime $L$, then its movement

$$\{X(t) : \tau \leq t < \tau + L\}$$

is distributed as

$$\{x + \eta(t - \tau) : \tau \leq t < \tau + L\}$$

where $\{\eta(\cdot)\}$ is a real-valued Markov process on $[0, \infty)$ with $\eta(0) = 0$. 
Scaling Limits of B-H processes with age dependent Markov motion

Assume that, for each individual, the lifetime $L$, the number of offspring $\xi$ and the movement process $\eta(\cdot)$ are independent and the triplets $(L, \xi, \eta)$ over all the individuals in the tree are i.i.d.

Let $Z_t$ be the population size at time $t$ and

$$C_t \equiv \{(a_{t,i}, x_{t,i}) : 1 \leq i \leq Z_t\}$$

be the age and position configuration of all the individuals alive at time $t$.

The object of study is the point process $\{C_t : t \geq 0\}$. 
Theorem 6.1

(Supercritical case) (Athreya-Athreya-Iyer, Bernoulli 2011)

Let \( p_0 = 0, 1 < m \equiv \sum_{j=1}^{\infty} j p_j < \infty \). Let \( E\eta(0) = 0 \),
\[
v(t) \equiv E\eta^2(t) < \infty, \quad \sup_{0 \leq s \leq t} v(s) < \infty \quad \text{and}
\]
\[
\psi_\alpha \equiv \int_{[0,\infty)} e^{-\alpha s} v(s) dG(s) < \infty
\]

where \( 0 < \alpha < \infty \) is the Malthusian parameter defined by
\[
m \int_{[0,\infty)} e^{-\alpha s} dG(s) = 1.
\]
Theorem 6.1

(continued) Let \((a_t, X_t)\) be the age and position of a randomly chosen individual at time \(t\). Then

a) \[
\left( a_t, \frac{X_t}{\sqrt{t}} \right) \xrightarrow{d} (U, V)
\]

where \(U\) and \(V\) are independent and \(U\) has pdf proportional to \(e^{-\alpha x} \left(1-G(x)\right)\) on \((0, \infty)\) and \(V\) is \(N\left(0, \frac{\psi_{\alpha}}{\mu_{\alpha}}\right)\) where

\[
\mu_{\alpha} = m \int_{0}^{\infty} xe^{-\alpha x} dG(x).
\]
Theorem 6.1

(continued)

b) Let

\[ Y_y(A \times B) = \frac{1}{Z_t} \sum_{i=1}^{Z_t} I_{A \times B}(a_{t,i}, \frac{x_{t,i}}{\sqrt{t}}) \]

be the scaled empirical measure of \( C_t \equiv \{(a_{t,i}, x_{t,i}) : 1 \leq i \leq Z_t\} \).

Then, \( Y_t \xrightarrow{d} (U, V) \), where \( U \) and \( V \) are as in a).

The proof of this depends on the following results of independent interest.

K. B. Athreya
Proposition 6.1

Let $M_t$ be the generation number of a randomly chosen individual from $Z_t$ (those alive at time $t$). Let \{\(L_{t,i}: 1 \leq t \leq M_t\}\} be the lifetimes of the ancestors of this individual. Then

a) as $t \to \infty$,

\[
\frac{M_t}{t} \to \frac{1}{\mu_\alpha} \quad \text{w.p. 1.}
\]
Proposition 6.1

Proposition (continued)

b) for any $h : [0, \infty) \to \mathbb{R}$ Borel measurable and
\[
\int_{[0,\infty)} |h(x)| e^{-\alpha x} dG(x) < \infty, \ 0 < \alpha < \infty,
\]
\[
P\left(\left| \frac{1}{M_t} \sum_{i=1}^{M_t} h(L_{t,i} - c_\alpha(h)) \right| > \epsilon \right) \to 0 \quad \text{as } t \to \infty.
\]

where $c_\alpha(h) = m \int_{[0,\infty)} h(x) e^{-\alpha x} dG(x)$. 

K. B. Athreya
Both these results depend on a size-biasing estimate of a large deviation result, namely,

**Proposition 6.2**

Let \( \{N(t) : t \geq 0\} \) be a renewal process generated by \( G \). Let \( 1 < m < \infty \) and \( \alpha \) be the Malthusian parameter, i.e.,

\[
m \int_{[0,\infty)} e^{-\alpha x} dG(x) = 1.
\]

Then, for \( \forall \epsilon > 0 \),

\[
e^{-\alpha t} E \left( m^{N(t)} I \left( \left| \frac{N(t)}{t} - \frac{1}{\mu_\alpha} \right| > \epsilon \right) \right) = 0
\]

where \( \mu_\alpha = m \int_0^\infty x e^{-\alpha x} dG(x) \).
Proposition 6.2

Note that since
\[ \frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{w.p.} 1 \]

where \( \mu = \int_{[0, \infty)} x dG(x) \), the event

\[ \left| \frac{N(t)}{t} - \frac{1}{\mu} \right| > \epsilon \]

is an event of large deviation.
Proposition 6.3

(Proposition)

(Coalescence time for BH process) (Athreya-Hong, 2011)
Choose two individuals from those alive at time \( t \) at random by SRSWOR and trace their lines back in time to find the time of death \( \tau_{t,2} \) of their last common ancestor. Let \( p_0 = 0 \),

\[
1 < m = \sum_{j=1}^{\infty} jp_j < \infty. \text{ Then, for } 0 < s < \infty, \\
\lim_{t \to \infty} P(\tau_{t,2} < s) = H(s) \quad \text{exists}
\]

and \( H(\cdot) \) is an absolutely continuous d. f. on \((0, \infty)\) with \( H(0) = 0, \ H(\infty) = 1. \)
Proposition 6.3

Same is true for the coalescence of $r$ individuals chosen at random from those alive at time $t$ (for $1 < r < \infty$).

However, the coalescence time for the whole population goes back to the beginning.

Open problems: Extend the results of Theorem 5.2 (BRW with $m = \infty$, $\{p_j\}_{j \geq 0} \in D(\alpha)$, $0 < \alpha < 1$) to the present setting.
Theorem 6.2

(Critical case) Let $m = 1$, $\sum_{j=1}^{\infty} j^2 p_j < \infty$, $E\eta(t) \equiv 0$, $v(t) = E\eta^2(t) < \infty$, $\sup_{0 \leq s \leq t} v(s) < \infty$, $\forall t$, and

$$\psi = \int_{[0, \infty)} v(s) dG(s) < \infty.$$ 

Let $A_t \equiv \{Z_t > 0\}$. Then,
Theorem 6.2

(continued) conditioned on $A_t$, the random vector

$$
\left( a_t, \frac{X_t}{t} \right)
$$

for a randomly chosen individual converges as $t \to \infty$ in distribution to $(U, V)$ where $U$ and $V$ are independent with $U$

having a pdf $\frac{1}{\mu} \left( 1 - G(\cdot) \right)$ on $(0, \infty)$ and $V \sim N\left(0, \frac{\psi}{\mu}\right)$. 

K. B. Athreya
Theorem 6.3

Assume the hypothesis of Theorem 6.2. Then, conditioned on $A_t \equiv \{Z_t > 0\}$, the empirical measure

$$Y_t(A \times B) \equiv \frac{1}{Z_t} \sum_{i=1}^{Z_t} I_{A \times B}(a_{t,i}, \frac{X_{t,i}}{\sqrt{t}})$$

converges as $t \to \infty$ in distribution to a random measure $\nu$ characterized by its moment sequence

$$m_k(\varphi) \equiv E(\langle \nu, \varphi \rangle)^k$$

where $\varphi \in C^+_b(\mathbb{R}^+ \times \mathbb{R})$. 
Theorem 6.3

The $m_k(\varphi)$ can be expressed in terms of the coalescence times of $k$ randomly chosen individuals alive at time $t$.

The proof depends on the following results.
Proposition 6.4

Let $m = 1$, $\sum_{j=1}^{\infty} j^2 p_j < \infty$, $G(\cdot)$ non-lattice. Then

i) $\forall \epsilon > 0$

$$P\left( \left| \frac{M_t}{t} - \frac{1}{\mu} \right| > \epsilon \left| Z_t > 0 \right. \right) \to 0 \quad \text{as } t \to \infty.$$
Proposition 6.4

(continued)

ii) the coalescence time \( \tau_{2,t} \) of two randomly chosen individuals from time \( t \) (conditioned on \( Z_t > 0 \)) satisfies

\[
\lim_{t \to \infty} P\left( \frac{\tau_{2,t}}{t} \leq x \left| Z_t > 0 \right. \right) = H(x) \quad \text{exists}
\]

for all \( 0 \leq x \leq 1 \).

iii) A similar result for the convergence of coalescence of \( k \) individuals.
Remark

Note that, in the supercritical case \((1 < m < \infty, p_0 = 0)\) BH process, \(\tau_{2,t}\) converged to a proper distribution as \(t \to \infty\).

And, in the critical case, \(\frac{\tau_{2,t}}{t}\) conditioned on \(Z_t > 0\) converges in distribution. That is, \(\tau_{2,t}\) is of order \(t\).

Related work: Lambert, Legall
References

1. Athreya, K. B. (2011), Coalescence in recent past in rapidly growing populations. (Submitted)


References


Thank You