Analogues of the Wiener-Tauberian and Schwartz theorems for radial functions on symmetric spaces

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Abstract

We prove a Wiener-Tauberian theorem for the $L^1$ spherical functions on a semisimple Lie group of arbitrary real rank. We also establish a Schwartz type theorem for complex groups. As a corollary we obtain a Wiener-Tauberian type result for compactly supported distributions.

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1 Introduction

Two celebrated theorems from classical analysis dealing with translation invariant subspaces are the Wiener-Tauberian theorem and the Schwartz theorem. Let $f \in L^1(\mathbb{R})$ and $\hat{f}$ be its Fourier transform. Then the celebrated Wiener-Tauberian theorem says that the ideal generated by $f$ is dense in $L^1(\mathbb{R})$ if and only if $\hat{f}$ is a nowhere vanishing function on the real line.

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The result due to L. Schwartz says that, every closed translation invariant subspace $V$ of $C^\infty(\mathbb{R})$ is generated by the exponential polynomials in $V$. In particular, such a $V$ contains the function $x \to e^{i\lambda x}$ for some $\lambda \in \mathbb{C}$. Interestingly, this result fails for $\mathbb{R}^n$, if $n \geq 2$. Even though an exact analogue of the Schwartz theorem fails for $\mathbb{R}^n \ n \geq 2$, it follows from the well known theorem of Brown-Schreiber-Taylor [BST] that, if $V \subset C^\infty(\mathbb{R}^n)$ is a closed subspace which is translation and rotation invariant then $V$ contains a $\psi_s$ for some $s \in \mathbb{C}$ where

$$\psi_s(x) = \frac{J_{\frac{n}{2}-1}(s|x|)}{(s|x|)^{\frac{n}{2}-1}} = \int_{S^{n-1}} e^{izx,w} d\sigma(w).$$

Here $J_{\frac{n}{2}-1}$ is the Bessel function of the first kind and order $n/2-1$ and $\sigma$ is the unique, normalized rotation invariant measure on the sphere $S^{n-1}$. It also follows from the work in [BST] that $V$ contains all the exponentials $e^{iz.x}$, if $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ satisfies $z_1^2 + z_2^2 + \cdots + z_n^2 = s^2$.

Our aim in this paper is to prove analogues of these results in the context of non compact semisimple Lie groups.

**Notation and preliminaries:** For any unexplained terminology we refer to [H]. Let $G$ be a connected non compact semisimple Lie group with finite center and $K$ a fixed maximal compact subgroup of $G$. Fix an Iwasawa decomposition $G = KAN$ and let $a$ be the Lie algebra of $A$. Let $a^*$ be the real dual of $a$ and $a^{*\mathbb{C}}$ its complexification. Let $\rho$ be the half sum of positive roots for the adjoint action of $a$ on $g$, the Lie algebra of $G$. The Killing form induces a positive definite form $<.,.>$ on $a^* \times a^*$. Extend this form to a bilinear form on $a^{*\mathbb{C}}$. We will use the same notation for the extension as well. Let $W$ be the Weyl group of the symmetric space $G/K$. Then there is a natural action of $W$ on $a, a^*, a^{*\mathbb{C}}$ and $<.,.>$ is invariant under this action.
For each $\lambda \in a^*_\mathfrak{g}$, let $\varphi_\lambda$ be the elementary spherical function associated with $\lambda$. Recall that $\varphi_\lambda$ is given by the formula
\[
\varphi_\lambda(x) = \int_K e^{(i\lambda - \rho)(H(xk))} \, dk \quad x \in G.
\]
See [H] for more details. It is known that $\varphi_\lambda = \varphi_{\lambda'}$ if and only if $\lambda' = \tau\lambda$ for some $\tau \in W$. Let $l$ be the dimension of $\mathfrak{a}$ and $F$ denote the set (in $\mathfrak{C}^l$)
\[
F = \mathfrak{a}^* + iC_\rho \quad \text{where } C_\rho = \text{convex hull of } \{s\rho : s \in W\}.
\]
Then it is a well known theorem of Helgason and Johnson that $\varphi_\lambda$ is bounded if and only if $\lambda \in F$.

Let $I(G)$ be the set of all complex valued spherical functions on $G$, that is
\[
I(G) = \{f : f(k_1xk_2) = f(x) : k_1, k_2 \in K, x \in G\}.
\]
Fix a Haar measure $dx$ on $G$ and let $I_1(G) = I(G) \cap L^1(G)$. Then it is well known that $I_1(G)$ is a commutative Banach algebra under convolution and that the maximal ideal space of $I_1(G)$ can be identified with $F/W$.

For $f \in I_1(G)$, define its spherical Fourier transform, $\hat{f}$ on $F$ by
\[
\hat{f}(\lambda) = \int_G f(x) \varphi_{-\lambda}(x) \, dx.
\]
Then $\hat{f}$ is a $W$ invariant bounded function on $F$ which is holomorphic in the interior $F^0$ of $F$, and continuous on $F$. Also $\widehat{f \ast g} = \hat{f} \hat{g}$ where the convolution of $f$ and $g$ is defined by
\[
f \ast g(x) = \int_G f(xy^{-1}) \, g(y) \, dy.
\]
Next, we define the $L^1$- Schwartz space of $K$-biinvariant functions on $G$ which will be denoted by $S(G)$. Let $x \in G$. Then $x = k \exp X$, $k \in K$, $X \in \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.
is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$. Put $\sigma(x) = \|X\|$, where $\|\cdot\|$ is the norm on $\mathfrak{p}$ induced by the Killing form. For any left invariant differential operator $D$ on $G$ and any integer $r \geq 0$, we define for a smooth $K$-biinvariant function $f$

$$p_{D,r}(f) = \sup_{x \in G} (1 + \sigma(x))^r |\varphi_0(x)|^{-2} |Df(x)|$$

where $\varphi_0$ is the elementary spherical function corresponding to $\lambda = 0$. Define

$$S(G) = \{ f : p_{D,r}(f) < \infty \text{ for all } D, r \}.$$

Then $S(G)$ becomes a Fréchet space when equipped with the topology induced by the family of semi norms $p_{D,r}$.

Let $P = P(\mathfrak{a}_Q^*)$ be the symmetric algebra over $\mathfrak{a}_Q^*$. Then each $u \in P$ gives rise to a differential operator $\partial(u)$ on $\mathfrak{a}_Q^*$. Let $Z(F)$ be the space of functions $f$ on $F$ satisfying the following conditions:

(i) $f$ is holomorphic in $F^0$ (interior of $F$) and continuous on $F$,

(ii) If $u \in P$ and $m \geq 0$ is any integer, then

$$q_{u,m}(f) = \sup_{\lambda \in F^0} (1 + \|\lambda\|^2)^m \left| \partial(u)f(\lambda) \right| < \infty,$$

(iii) $f$ is $W$ invariant.

Then $Z(F)$ is an algebra under pointwise multiplication and a Fréchet space when equipped with the topology induced by the seminorms $q_{u,m}$.

If $a \in Z(F)$ we define the “wave packet” $\psi_a$ on $G$ by

$$\psi_a(x) = \frac{1}{|W|} \int_{\mathfrak{a}_r^*} a(\lambda) \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda,$$

where $c(\lambda)$ is the well known Harish-Chandra $c$-function. By the Plancherel theorem due to Harish-Chandra we also know that the map $f \to \hat{f}$ extends to a unitary map
from $L^2(K\backslash G/K)$ onto $L^2(a^*, |c(\lambda)|^{-2}d\lambda)$. We are now in a position to state a result of Trombi-Varadarajan [TV].

**Theorem 1.1**

(i) If $f \in S(G)$ then $\hat{f} \in Z(F)$.

(ii) If $a \in Z(F)$ then the integral defining the “wave packet” $\psi_a$ converges absolutely and $\psi_a \in S(G)$. Moreover, $\hat{\psi}_a = a$.

(iii) The map $f \to \hat{f}$ is a topological linear isomorphism of $S(g)$ onto $Z(F)$.

The plan of this paper is as follows: in the next section we prove a Wiener-Tauberian theorem for $L^1(K\backslash G/K)$ assuming more symmetry on the generating family of functions. In the final section we establish a Schwartz type theorem for complex semisimple Lie groups. As a corollary we also obtain a Wiener-Tauberian type theorem for compactly supported distributions on $G/K$.

## 2 A Wiener-Tauberian theorem for $L^1(K\backslash G/K)$

In [EM], Ehrenpreis and Mautner observed that an exact analogue of the Wiener-Tauberian theorem is not true for the commutative algebra of $K$-biinvariant functions on the semisimple Lie group $SL(2, \mathbb{R})$. Here $K$ is the maximal compact subgroup $SO(2)$. However, in the same paper it was also proved that an additional “not too rapidly decreasing condition” on the spherical Fourier transform of a function suffices to prove an analogue of the Wiener-Tauberian theorem. That is, if $f$ is a $K$-biinvariant integrable function on $G = SL(2, \mathbb{R})$ and its spherical Fourier transform $\hat{f}$ does not vanish anywhere on the maximal ideal space (which can be identified with a certain strip on the complex plane) then the function $f$ generates a dense subalgebra of $L^1(K\backslash G/K)$ provided $\hat{f}$ does not vanish too fast at $\infty$. See [EM] for precise statements.
There have been a number of attempts to generalize these results to $L^1(K\backslash G/K)$ or $L^1(G/K)$ where $G$ is a non compact connected semisimple Lie group with finite center. Almost complete results have been obtained when $G$ is a real rank one group. We refer the reader to [BW], [BBHW] [RS98] and [S88] for results on rank one case. See also [RS97] for a result on the whole group $SL(2, \mathbb{R})$.

In [S80], it is proved that under suitable conditions on the spherical Fourier transform of a single function $f$ an analogue of the Wiener-Tauberian theorem holds for $L^1(K\backslash G/K)$, with no assumptions on the rank of $G$. Recently, the first named author improved this result to include the case of a family of functions rather than a single function (see [N]). One difference between rank one results and higher rank results has been the precise form of the “not too rapid decay condition”. In [S80] and [N] this condition on the spherical Fourier transform of a function is assumed to be true on the whole maximal domain, while for rank one groups it suffices to have this condition on $a^*$ (see [BW] and [RS98]) (An important corollary of this is that, in the rank one case one can get a Wiener-Tauberian type theorem for a wide class of functions purely in terms of the non vanishing of the spherical Fourier transform in a certain domain without having to check any decay conditions, see [MRSS], Theorem 5.5). In the first part of this paper we show that such a stronger result is true for higher rank case as well provided we assume more symmetry on the generating family of functions, and again as a corollary we get a result of the type alluded to in the parenthesis above.

If $\dim a^* = l$, then $a^*_G$ may be identified with $\mathbb{C}^l$ and a point $\lambda \in a^*_G$ will be denoted $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. Let $B_R$ denote the ball of radius $R$ centered at the origin in $a^*$ and $F_R$ denote the domain in $a^*_G$, defined by

$$F_R = \{ \lambda \in a^*_G : \|Im(\lambda)\| < R\}.$$
For $a > 0$, let $I_a$ denote the strip in the complex plane defined by

$$I_a = \{ z \in \mathbb{C} : |Imz| < a \}.$$

Now, suppose that $f$ is a holomorphic function on $F_R$ and $f$ depends only on $(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda^2_l)^{\frac{1}{2}}$. Then it is easy to see that

$$g(s) = f(\lambda_1, \lambda_2, \ldots, \lambda_l)$$

where $s^2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2$ defines an even holomorphic function on $I_R$ and vice versa.

We will need the following lemmas. Let $A(I_a)$ denote the collection of functions $g$ with the properties:

(i) $g$ is even, bounded and holomorphic on $I_a$,
(ii) $g$ is continuous on $\overline{I}_a$,
(iii) $\lim_{|s| \to \infty} g(s) = 0$.

Then $A(I_a)$ with the supremum norm is a Banach algebra under pointwise multiplication.

**Lemma 2.1** Let $\{g_\alpha : \alpha \in I\}$ be a collection of functions in $A(I_a)$. Assume that there exists no $s \in \overline{I}_a$ such that $g_\alpha(s) = 0 \forall \alpha \in I$. Further assume that there exists $\alpha_0 \in I$ such that $g_{\alpha_0}$ does not decay very rapidly on $R$, i.e,

$$\limsup_{|s| \to \infty} |g_{\alpha_0}(s)| e^{ke^{k|s|}} > 0$$

on $R$ for all $k > 0$. Then the closed ideal generated by $\{g_\alpha : \alpha \in I\}$ is whole of $A(I_a)$.

**Proof:** Let $\psi$ be a suitable biholomorphic map which maps the strip $I_a$ onto the unit disc (see [BW]). Let $h_\alpha(z) = g_\alpha(\psi(z))$. Then $h_\alpha \in A_0(D)$, where $A_0(D)$ is the
collection of even holomorphic functions \( h \) on the unit disc, continuous up to the boundary and \( h(i) = h(-i) = 0 \). The not too rapid decay condition on \( IR \) is precisely what is needed to apply the Beurling-Rudin theorem to complete the proof. We refer to [BW] (see the proof of Theorem 1.1 and Lemma 1.2) for the details.

Let \( p_t \) denote the \( K \)-biinvariant function defined by \( \hat{p}_t(\lambda) = e^{-t\langle \lambda, \lambda \rangle} \). It is easy to see that \( p_t \in S(G) \).

**Lemma 2.2** Let \( J \subset L^1(K\backslash G/K) \) be a closed ideal. If \( p_t \in J \) for some \( t > 0 \), then \( J = L^1(K\backslash G/K) \).

**Proof:** This follows from the main result in [N] or [S80].

Before we state our main theorem we define the following: We say that a function \( f \in L^1(K\backslash G/K) \) is **radial** if the spherical Fourier transform \( \hat{f}(\lambda) \) is a function of \((\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}}\). Notice that, if the group \( G \) is of real rank one, then the class of radial functions is precisely the class of \( K \)-biinvariant functions in \( L^1(G) \). When the group \( G \) is complex, it is possible to describe the class of radial functions (see next section). The following is our main theorem in this section:

**Theorem 2.3** Let \( \{f_\alpha : \alpha \in I\} \) be a collection of radial functions in \( L^1(K\backslash G/K) \). Assume that the spherical transform \( \hat{f}_\alpha \) extends as a bounded holomorphic function to the bigger domain \( F_R \), where \( R > \|\rho\| \) with \( \lim_{|\lambda| \to \infty} \hat{f}_\alpha(\lambda) = 0 \) for all \( \alpha \) and that there exists no \( \lambda \in F_R \) such that \( \hat{f}_\alpha(\lambda) = 0 \) for all \( \alpha \). Further assume that there exists an \( \alpha_0 \) such that \( \hat{f}_{\alpha_0} \) does not decay too rapidly on \( a^* \), i.e,

\[
\limsup_{|\lambda| \to \infty} |\hat{f}_{\alpha_0}(\lambda)| \exp(k|\lambda|) > 0
\]

for all \( k > 0 \) on \( a^* \). Then the closed ideal generated by \( \{f_\alpha : \alpha \in I\} \) is all of \( L^1(K\backslash G/K) \).
Proof: Since \( f_{\alpha} \) is radial, each \( \hat{f}_{\alpha} \) gives rise to an even bounded holomorphic function \( g_{\alpha}(s) \) on the strip \( I_R \). If \(|\rho| < a < R\), then the collection \( \{ g_{\alpha}(s), \alpha \in I \} \) satisfies the hypotheses in Lemma 2.1 on the domain \( I_a \). It follows that the family \( \{ g_{\alpha} \} \) generates \( A(I_a) \). In particular, we have a sequence

\[
h_1^n(s)g_{\alpha_1(n)}(s) + h_2^n(s)g_{\alpha_2(n)}(s) + \cdots + h_k^n(s)g_{\alpha_k(n)}(s) \to e^{-\frac{s^2}{2}}
\]

uniformly on \( \overline{I}_a \), where \( g_{\alpha_j(n)} \) are in the given family and \( h_j^n(s) \in A(I_a) \).

Notice that each \( h_j^n \) can be viewed as a holomorphic function on the domain \( F_a \) contained in \( a^{*}_{\mathbb{Q}^p} \) which depends only on \( (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}} \). Since \( h_j^n \) are bounded and \(|\rho| < a\) it can be easily verified that \( e^{-\frac{(\lambda,\lambda)}{2}}h_j^n(\lambda) \in Z(F) \). Again, an application of the Cauchy integral formula says that

\[
e^{-\frac{(\lambda,\lambda)}{2}}h_1^n(\lambda)\hat{f}_{\alpha_1(n)}(\lambda) + e^{-\frac{(\lambda,\lambda)}{2}}h_2^n(\lambda)\hat{f}_{\alpha_2(n)}(\lambda) + \cdots + e^{-\frac{(\lambda,\lambda)}{2}}h_k^n(\lambda)\hat{f}_{\alpha_k(n)}(\lambda)
\]

converges to \( e^{-\langle \lambda,\lambda \rangle} \) in the topology of \( Z(F) \) (see the proof of Theorem 1.1 in [BW]).

By Theorem 1.1 this simply means that the ideal generated by \( \{ f_{\alpha} : \alpha \in I \} \) in \( L^1(K \backslash G/K) \) contains the function \( p \) where \( \hat{p}(\lambda) = e^{-\langle \lambda,\lambda \rangle} \). We finish the proof by appealing to Lemma 2.2.

**Corollary 2.4** Let \( \{ f_{\alpha} : \alpha \in I \} \) be a family of radial functions satisfying the hypotheses in Theorem 2.3. Then the closed subspace spanned by the left \( G \)-translates of the above family is all of \( L^1(G/K) \).

**Proof:** Let \( J \) be the closed subspace generated by the left translates of the given family. By Theorem 2.3, \( L^1(K \backslash G/K) \subset J \). Now, it is easy to see that \( J \) has to be equal to \( L^1(G/K) \).
Corollary 2.5 Let \( \{ f_\alpha : \alpha \in I \} \) be a family of \( L^1 \)--radial functions. Assume that each \( \hat{f}_\alpha \) extends to a bounded holomorphic function to the bigger domain \( F_R \) for some \( R > \| \rho \| \). Assume further that \( \lim_{\| \lambda \| \to \infty} \hat{f}_\alpha(\lambda) \to 0 \). If there exists an \( \alpha_0 \) such that \( f_{\alpha_0} \) is not equal to a real analytic function almost everywhere, then the left \( G \)--translates of the above family span a dense subset of \( L^1(G/K) \).

**Proof:** This follows exactly as in Theorem 5.5 of [MRSS].

### 3 Schwartz theorem for complex groups

When \( G \) is a connected non compact semisimple Lie group of real rank one with finite center, a Schwartz type theorem was proved by Bagchi and Sitaram in [BS79]. Let \( K \) be a maximal compact subgroup of \( G \), then the result in [BS79] states the following: Let \( V \) be a closed subspace of \( C^\infty(K \backslash G/K) \) with the property that \( f \in V \) implies \( w*f \in V \) for every compactly supported \( K \)--biinvariant distribution \( w \) on \( G/K \), then \( V \) contains an elementary spherical function \( \varphi_\lambda \) for some \( \lambda \in \mathfrak{a}^*_C \). This was done by establishing a one-one correspondence between ideals in \( C^\infty(K \backslash G/K) \) and that of \( C^\infty(\mathbb{R})_{even} \). This also proves that a similar result can not hold for higher rank groups.

Going back to \( \mathbb{R}^n \), we notice that if \( f \in C^\infty(\mathbb{R}^n) \) is radial, then the translation invariant subspace \( V_f \) generated by \( f \) is also rotation invariant. It follows from [BST] that \( V_f \) contains a \( \psi_s \) for some \( s \in \mathfrak{c} \) where \( \psi_s \) is the Bessel function defined in the introduction. Our aim in this section is to prove a similar result for the complex semisimple Lie groups. Our definition of *radiality* is taken from [VV] and it coincides with the definition in the previous section when the function is in \( L^1(K \backslash G/K) \).
Throughout this section we assume that $G$ is a complex semisimple Lie group. Let $\text{Exp} : \mathfrak{p} \to G/K$ denote the map $P \to (\exp P)K$. Then $\text{Exp}$ is a diffeomorphism. If $dx$ denotes the $G$–invariant measure on $G/K$, then

$$\int_{G/K} f(x) \, dx = \int_{\mathfrak{p}} f(\text{Exp} P) \, J(P) \, dP,$$

where

$$J(P) = \det \left( \frac{\sinh \text{ad} P}{\text{ad} P} \right).$$

Since $G$ is a complex group, the elementary spherical functions are given by a simple formula:

$$\varphi_\lambda(\text{Exp} P) = J(P)^{-\frac{1}{2}} \int_K e^{i(A_\lambda, \text{Ad}(k)P)} \, dk, \quad P \in \mathfrak{p}.$$  \hfill (3.2)

Here $A_\lambda$ is the unique element in $\mathfrak{a}_G^\mathbb{C}$ such that $\lambda(H) = \langle A, A_\lambda \rangle$ for all $H \in \mathfrak{a}_G^\mathbb{C}$.

Let $E(K\backslash G/K)$ be the dual of $C^\infty(K\backslash G/K)$. Then $E(K\backslash G/K)$ can be identified with the space of compactly supported $K$-biinvariant distributions on $G/K$. If $w$ is such a distribution then $\hat{w}(\lambda) = w(\varphi_\lambda)$ is well defined and is called the spherical Fourier transform of $w$. By the Paley-Wiener theorem we know that $\lambda \to \hat{w}(\lambda)$ is an entire function of exponential type. Similarly, $E(\mathbb{R}^l)$ will denote the space of compactly supported distribution on $\mathbb{R}^l$ and $E^W(\mathbb{R}^l)$ consists of the Weyl group invariant ones. From the work in [BS79] we know that the Abel transform

$$S : E(K\backslash G/K) \to E^W(\mathbb{R}^l)$$

is an isomorphism and $\hat{S}(w)(\lambda) = \hat{w}(\lambda)$ for $w \in E(K\backslash G/K)$, where $\hat{S}(w)(\lambda)$ is the Euclidean Fourier transform of the distribution $S(w)$. We also need the following result from [BS79].
Proposition 3.1 There exists a linear topological isomorphism $T$ from $C^\infty(K \backslash G/K)$ onto $C^\infty(\mathbb{R}^l)^W$ such that

$$S(w)(T(f)) = w(f)$$

for all $w \in E(K \backslash G/K)$ and $f \in C^\infty(K \backslash G/K)$. We also have,

$$S(w') * T(w * f) = T(w' * w * f)$$

for all $w, w' \in E(K \backslash G/K)$ and $f \in C^\infty(K \backslash G/K)$. Moreover,

$$T(\varphi_\lambda) = \frac{1}{|W|} \sum_{\tau \in W} \exp(i(\tau, \lambda, x))$$

A $K$-biinvariant function $f$ is called radial if it is of the form

$$f(x) = J(Exp^{-1}x)^{-\frac{1}{2}}u(d(0,x)),$$

where $d$ is the riemannian distance induced by the the Killing form on $G/K$ and $u$ is a function on $[0, \infty)$. Theorem 4.6 in [VV] shows that this definition of radiality coincides with the one in the previous section if the function is integrable. That is, $f \in L^1(K \backslash G/K)$ has the above form if and only if the spherical Fourier transform $\hat{f}(\lambda)$ depends only on $(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}}$. We denote the class of smooth radial functions by $C^\infty(K \backslash G/K)_rad$ and $C^\infty_c(K \backslash G/K)_rad$ will consists of compactly supported functions in $C^\infty(K \backslash G/K)_rad$.

For $f \in C^\infty(K \backslash G/K)$ define

$$f^\#(ExpP) = J(P)^{\frac{1}{2}} \int_{SO(p)} J(\sigma P)^{\frac{1}{2}} f(\sigma P) \, d\sigma,$$

where $SO(p)$ is the special orthogonal group on $p$ and $d\sigma$ is the Haar measure on $SO(p)$. Here, by $f(P)$ we mean $f(ExpP)$. Clearly, $f \to f^\#$ is the projection from $C^\infty(K \backslash G/K)$ onto $C^\infty(K \backslash G/K)_rad$. 

12
Proposition 3.2  (a) The space $C^\infty(K\backslash G/K)_{\text{rad}}$ is reflexive.

(b) The strong dual $E(K\backslash G/K)_{\text{rad}}$ of $C^\infty(K\backslash G/K)_{\text{rad}}$ is given by

$$\{ w \in E(K\backslash G/K) : \hat{w}(\lambda) \text{ is a function of } (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}} \}. $$

(c) The space $C^\infty(K\backslash G/K)_{\text{rad}}$ is invariant under convolution by $w \in E(K\backslash G/K)_{\text{rad}}$.

Proof: (a) The space $C^\infty(K\backslash G/K)_{\text{rad}}$ is a closed subspace of $C^\infty(K\backslash G/K)$ which is a reflexive Fréchet space.

(b) Define $B_\lambda = \varphi^\#_\lambda$, the projection of $\varphi_\lambda$ into $C^\infty(K\backslash G/K)_{\text{rad}}$. A simple computation shows that

$$B_\lambda(\text{ExpP}) = J(P)^{-\frac{1}{2}} \int_{SO(p)} e^{ij(A_\lambda, \sigma P)} d\sigma.$$ 

It is clear that, $B_\lambda$ as a function of $\lambda$ depends only on $(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}}$. Now, let $w \in E(K\backslash G/K)$. Define a distribution $w^\#$ by $w^\#(f) = w(f^\#)$. It is easy to see that $w^\#$ is a compactly supported $K$-biinvariant distribution. Clearly, if $w \in E(K\backslash G/K)_{\text{rad}}$, then $w = w^\#$. It follows that $\hat{w}(\lambda) = w(\varphi_\lambda) = w(B_\lambda)$. Consequently, $\hat{w}(\lambda)$ is a function of $(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}}$. It also follows that $E(K\backslash G/K)_{\text{rad}}$ is reflexive.

(c) Observe that if $w \in E(K\backslash G/K)_{\text{rad}}$ and $g \in C^\infty_c(K\backslash G/K)_{\text{rad}}$ then $w * g \in C^\infty_c(K\backslash G/K)_{\text{rad}}$. This follows from (b) above and Theorem 4.6 in [VV]. Next, if $g$ is arbitrary, we may approximate $g$ with $g_n \in C^\infty_c(K\backslash G/K)_{\text{rad}}$.

We are in a position to state our main result in this section. Let $V$ be a closed subspace of $C^\infty_c(K\backslash G/K)_{\text{rad}}$. We say, $V$ is an ideal in $C^\infty(K\backslash G/K)_{\text{rad}}$ if $f \in V$ and $w \in E(K\backslash G/K)_{\text{rad}}$ implies that $w * f \in V$.

Theorem 3.3  (a) If $V$ is a non zero ideal in $C^\infty(K\backslash G/K)_{\text{rad}}$ then there exists a $\lambda \in \mathfrak{a}^*_q$, such that $B_\lambda \in V$. 

13
(b) If \( f \in C^\infty(K\backslash G/K)_{rad} \), then the closed left \( G \) invariant subspace generated by \( f \) in \( C^\infty(G/K) \) contains a \( \varphi_\lambda \) for some \( \lambda \in \mathfrak{a}_G^* \).

**Proof:** We closely follow the arguments in [BS79].

(a) Notice that the map

\[
S : E(K\backslash G/K)_{rad} \to E(\mathbb{R}^l)_{rad}
\]

is a linear topological isomorphism. Using the reflexivity of the spaces involved and arguing as in [BS79] we obtain that (as in Proposition 3.1)

\[
T : C^\infty(K\backslash G/K)_{rad} \to C^\infty(\mathbb{R}^l)_{rad}
\]

is a linear topological isomorphism, where \( C^\infty(\mathbb{R}^l)_{rad} \) stands for the space of \( C^\infty \) radial functions on \( \mathbb{R}^l \) and

\[
S(w)(T(f)) = w(f) \quad \forall w \in E(K\backslash G/K)_{rad}, f \in C^\infty(K\backslash G/K)_{rad}.
\]

Another application of Proposition 3.1 implies that we have a one-one correspondence between the ideals in \( C^\infty(K\backslash G/K)_{rad} \) and \( C^\infty(\mathbb{R}^l)_{rad} \). Here, ideal in \( C^\infty(\mathbb{R}^l)_{rad} \) means a closed subspace invariant under convolution by compactly supported radial distributions on \( \mathbb{R}^l \). From [BS90] or [BST] we know that any ideal in \( C^\infty(\mathbb{R}^l)_{rad} \) contains a \( \psi_s \) (Bessel function) for some \( s \in \mathcal{C} \). To complete the proof it suffices to show that under the topological isomorphism \( T \) the function \( B_\lambda \) is mapped into \( \psi_s \) where \( s^2 = (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^2 \).

Now, we have \( S(w)(T(B_\lambda)) = w(B_\lambda) \). Since \( w \in E(K\backslash G/K)_{rad} \) we know that \( w(B_\lambda) \) is nothing but \( w(\varphi_\lambda) \) which equals \( (S\overline{w})(\lambda) \). Since \( S \) is onto, this implies that \( T(B_\lambda) = \psi_s \) where \( s^2 = (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_l^2)^{\frac{1}{2}} \).
(b) From [BS79] we know that $T(\varphi_\lambda) = \psi_\lambda$ where $\psi_\lambda(x) = \frac{1}{|W|} \sum_{\tau \in W} \exp(i\tau \lambda x)$. Let $V_f$ denote the left $G$-invariant subspace generated by $f$. Then $T(V_f)$ surely contains the space

$$V_{T(f)} = \{ S(w) \ast T(f) : \ w \in E(K \backslash G/K) \}. $$

From Proposition 3.2, $T(f)$ is a radial $C^\infty$ function on $\mathbb{R}^l$. Hence, from [BST], the translation invariant subspace $X_{T(f)}$, generated by $T(f)$ in $C^\infty(\mathbb{R}^l)$ contains a $\psi_s$ for some $s \in a^*$ and consequently all the exponentials $e^{iz \cdot x}$ where $z = (z_1, z_2, \ldots, z_l)$ satisfies $z_1^2 + z_2^2 + \cdots + z_l^2 = s^2$. Now, it is easy to see that the map $g \rightarrow g^W$ where $g^W(x) = \frac{1}{|W|} \sum_{\tau \in W} g(\tau x)$, from $X_{T(f)}$ into $V_{T(f)}$ is surjective. Hence, there exists a $\lambda \in a^*$ such that $\psi_\lambda \in V_{T(f)}$. Since $T(\varphi_\lambda) = \psi_\lambda$, this finishes the proof.

Our next result is a Wiener-Tauberian type theorem for compactly supported distributions. Let $E(G/K)$ denote the space of compactly supported supported distributions on $G/K$. If $g \in G$ and $w \in E(G/K)$ then the left $g$-translate of $w$ is the compactly supported distribution $^gw$ defined by

$$^gw(f) = w(g^{-1} f), \ f \in C^\infty(G/K)$$

where $^x f(y) = f(x^{-1} y)$.

**Theorem 3.4** Let $\{w_\alpha : \alpha \in I\}$ be a family of distributions contained in $E(K \backslash G/K)_{rad}$. Then, the left $G$-translates of this family spans a dense subset of $E(G/K)$ if and only if there exists no $\lambda \in a^*_G$ such that $\hat{w}_\alpha(\lambda) = 0$ for all $\alpha \in I$.

**Proof:** We start with the *if* part of the theorem. Let $J$ stand for the closed span of the left $G$-translates of the distributions $w_\alpha$ in $E(G/K)$. It suffices to show that $E(K \backslash G/K) \subset J$. To see this, let $f \in C^\infty(G/K)$ be such that $w(f) = 0$ for all
\( w \in E(K \backslash G/K) \). Since \( J \) is left \( G \)-invariant we also have \( w(f_g) = 0 \) for all \( g \in G \), where \( f_g \) is the \( K \)-biinvariant function defined by

\[
f_g(x) = \int_K f(gkx) \, dk.
\]

It follows that \( f_g \equiv 0 \) for all \( g \in G \) and consequently \( f \equiv 0 \).

Next, we claim that if \( E(K \backslash G/K)_{rad} \subset J \) then \( E(K \backslash G/K) \subset J \). To prove this it is enough to show that

\[
\{ g \ast w : w \in E(K \backslash G/K)_{rad}, g \in C_c^\infty(K \backslash G/K) \}
\]

is dense in \( E(K \backslash G/K) \). Notice that, by Proposition 3.2 the map \( S \) from \( E(K \backslash G/K) \) onto \( E(\mathbb{R}^d)^W \) is a linear topological isomorphism which maps \( E(K \backslash G/K)_{rad} \) onto \( E(\mathbb{R}^d)_{rad} \) isomorphically. Hence, it suffices to prove a similar statement for \( E(\mathbb{R}^d)_{rad} \) and \( E(\mathbb{R}^d)^W \) which is an easy exercise in distribution theory!

So, to complete the proof of Theorem 3.4 we only need to show that

\[
\{ g \ast w_\alpha : \alpha \in I, g \in C_c^\infty(K \backslash G/K)_{rad} \}
\]

is dense in \( E(K \backslash G/K)_{rad} \). If not, consider

\[
J_{rad} = \{ f \in C^\infty(K \backslash G/K)_{rad} : (g \ast w_\alpha)(f) = 0 \ \forall g \in C_c^\infty(K \backslash G/K), \ \alpha \in I \}.
\]

The above is clearly a closed subspace of \( C^\infty(K \backslash G/K)_{rad} \) which is invariant under convolution by \( C_c^\infty(K \backslash G/K)_{rad} \). By Theorem 3.3 we have \( B_\lambda \in J_{rad} \) for some \( \lambda \in a_0^* \). It follows that \( \hat{w}_\alpha(\lambda) = 0 \) for all \( \alpha \in I \) which is a contradiction. This finishes the proof.

For the only if part, it suffices to observe that if \( g \in C_c^\infty(G/K) \) then

\[
g \ast w_\alpha(\varphi_\lambda) = \hat{g}(\lambda) \hat{w}_\alpha(\lambda)
\]

16
where \( g^\#(x) = \int_K g(kx) \, dk \).

**Remark:** A similar theorem for all rank one spaces may be derived from the results in [BS90].

### References


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